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Isospectral Deformations in QFT: The Massive Case

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Consideramos deformaciones isoespectrales de teorías cuánticas de campo usando como nueva herramienta de construcción a las convoluciones deformadas. La deformación nos permite obtener una variedad de modelos que son local en cuña y tienen matrices de dispersión no triviles.

We consider isospectral deformations of quantum field theories by using the novel construction tool of warped convolutions. The deformation enables us to obtain a variety of models that are wedge-local and have nontrivial scattering matrices.

PALABRAS CLAVES

TCC, TCC no conmutativa, Cuantización por deformación, Cuantización por deformación estricta, Convoluciones deformadas, Espacio-tiempos no conmutativos, Geometría no conmutativa

KEYWORDS

QFT, Non-commutative QFT, Deformation quantization, Strict deformation quantization, Warped convolutions, Non-commutative Space-times, Non-commutative Geometry

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I | INTRODUCTION

The equivalence of scalar fields living on a constant Moyal-Weyl space-time and wedge-local fields was proven in the ground breaking paper (Grosse & Lechner, 2007). This important insight has since its publication generated a vivid interest in algebraic and constructive QFT (Alazzawi, n.d.; Bostelmann & Cadamuro, 2013; Buchholz, Lechner, & Summers, 2011; Buchholz & Summers, n.d.; Grosse & Lechner, 2008; Lechner, 2012; Lechner, Schlemmer, & Tanimoto, 2013; Morfa-Morales, 2011; Much, 2012).

Wedge-local fields possess a weaker form of locality than point-like local fields. The localization is given on the wedges (see Section 2 for the exact definition). It is the approporiate generalization for noncommutative spacetimes since in those spacetimes the notion of a spacetime point is missing.

In this context an important tool has been formulated (Buchholz et al., 2011; Buchholz & Summers, n.d.), known as warped convolutions, in order to deform quantum fields in a rigorous mathematical fashion. This precise mathematical formulation of deformation theory was used to construct new QFT-models from a free theory at hand, (Alazzawi, n.d.; Bostelmann & Cadamuro, 2013; Lechner, 2012; Lechner et al., 2013; Morfa-Morales, 2011; Much, 2012). In particular the investigation shows that the newly obtained models have non-trivial scattering matrices, which even satisfy weakened relativistic locality and covariance properties. The weakened properties are interesting from a physical point of view since relativistic symmetries are hard to realize when the notion of a point (non-commutative geometry) is lost. The technique of warped convolutions has been as well used in a quantum mechanical context in order to obtain quantum mechanical effects and attack the quantum measurement problem, (Andersson, November 2013; Much, 2014).

To obtain scalar fields living on a constant Moyal-Weyl space-time by using warped convolutions, one uses as generator of deformation the momentum operator. Hence, fields deformed with the **momentum operator** correspond to wedge local fields which are equivalent to fields living on a constant Moyal-Weyl space-time. The question which is the subject of investigation in this paper is the following: Are there more wedge-local fields corresponding to excitations on non-commutative space-times that differ from the constant Moyal-Weyl space-time? An answer to this problem was given on the basis of a concrete example (Much, 2012), where the special conformal operators were used instead of the momentum operators for deformation. The proofs in the case of the conformal transformations used the unitary equivalence to the momentum operator. Is this program extendable? Hence, can we take operators that are unitary equivalent to the momentum operator and obtain wedge-local fields on one hand and excitations living on a nontrivial non-commutative space-time on the other hand? A detailed answer will be given in this paper. We first define the operators on the appropriate domains and investigate how far the program of wedge-locality can be achieved.

The investigation shows that the Wightman properties of scalar fields deformed with the unitary equivalent operator are satisfied without any restriction. However, concerning the wedge-locality we obtain an additional requirement on the operators used for deformation. Moreover, to use the concept of tempered polarization free generators (Borchers, Buchholz, & Schroer, 2001; Schroer, 1997) for scattering, we are obliged to show temperateness and polynomial boundedness of the field. The award of this concept is the ability to calculate an explicit two-particle scattering of the deformed theory. Next to investigating locality, covariance and scattering we study the relation of deformed fields to a non-commutative space-time. The first step in this direction is to construct an isomorphism from the deformed $^*-$algebras to the unitary transformed $^*-$algebras of fields living on Moyal-Weyl. By using simple examples it is further shown how the deformed $^*-$algebras relate to the twist-deformation framework, i.e. to fields defined on a non-commutative space-time.

The paper is organized as follows: First we lay out the novel tool of warped convolutions with all the definitions, lemmas and propositions needed for this work. In the third Section we define the operators used for deformation and prove that warped convolutions of the free scalar field, given as an oscillatory integral, are well-defined. Furthermore, the Wightman properties and wedge-locality are proven for a specific set of operators. The section ends with a whole class of examples. The fourth section describes the scattering of the constructed models. Last but not least we turn to the heart piece of this paper: The investigation of the resulting non-commutative space-times generated by isospectral deformations.

II | WARPED CONVOLUTIONS IN QFT

In this section we write all basic definitions and lemmas of the deformation known under the name of warped convolutions. For proofs of the respective lemmas we refer the reader to the original works (Buchholz et al., 2011; Grosse & Lechner, 2007).

The authors start their investigation with a C^*-dynamical system $(\mathcal{A},\mathbb{R}^d)$, Pedersen (1979). It consists of a C^*-algebra $\mathcal{A}$ equipped with a strongly continuous automorphic action of the group $\mathbb{R}^d$ which will be denoted by α. Furthermore, let $\mathcal{B}(H)$ be the Hilbert space of bounded operators on H and let the adjoint action α be implemented by the weakly continuous unitary representation U. Then, it is argued that since the unitary representation U can be extended to the algebra $\mathcal{B}(H)$, there is no lost of generality when one proceeds to the C^*-dynamical system $(C^* \subset \mathcal{B}(H),\mathbb{R}^d)$. Here $C^* \subset \mathcal{B}(H)$ is the C^*-algebra of all operators on which α acts strongly continuously.

Hence, we start by assuming the existence of a strongly continuous unitary group U that is a representation of the additive group $\mathbb{R}^d$, $d \geq 2$, on some separable Hilbert space H. Moreover, to define the deformation of operators belonging to a C^*-algebra $C^* \subset \mathcal{B}(H)$, we consider elements belonging to the sub-algebra $C^\infty \subset C^*$. The sub-algebra C^∞ is defined to be the $*-$algebra of smooth elements (in the norm topology) with respect to α, which is the adjoint action of a weakly continuous unitary representation U of $\mathbb{R}^d$ given by

$$\alpha_x(A) = U(x)AU(x)^{-1}, \quad x \in \mathbb{R}^d.$$

Let $\mathcal{D}$ be the dense domain of vectors in H which transform smoothly under the adjoint action of U. Then, the warped convolutions for operators $A \in C^\infty$ are given by the following definition.

Definition Let θ be a real skew-symmetric matrix relative to the chosen bilinear form on $\mathbb{R}^d$, let $A \in C^\infty$ and let E be the spectral resolution of the unitary operator U. Then, the corresponding warped convolution A_θ of A is defined on the domain $\mathcal{D}$ according to

$$A_\theta := \int \alpha_{\theta x}(A)dE(x), \tag{1}$$

where α denotes the adjoint action of U given by $\alpha_k(A) = U(k)AU(k)^{-1}$.

The restriction in the choice of operators is owed to the fact that the deformation is performed with operator valued integrals. Furthermore, one can represent the warped convolution of $A \in C^\infty$ by $\int \alpha_{\theta x}(A)dE(x)$ or $\int dE(x)\alpha_{\theta x}(A)$, on the dense domain $\mathcal{D} \subset H$ of vectors smooth w.r.t. the action of U, in terms of strong limits

$$\int \alpha_{\theta x}(A)dE(x)\Phi = (2\pi)^{-d} \lim_{\varepsilon\to 0} \iint dx\,dy\,\chi(\varepsilon x,\varepsilon y)\, e^{-ixy}\,\alpha_{\theta x}(A)\,U(y)\Phi,$$

where $\chi \in S(\mathbb{R}^d \times \mathbb{R}^d)$ with $\chi(0,0) = 1$. This representation makes calculations and proofs concerning the existence of integrals easier. In this work we use both representations.

The following lemma shows first that the two different warped convolutions are equivalent. Second, it shows how the complex conjugation acts on the warped convoluted operator.

Lemma 1. *Let* θ *be a real skew symmetric matrix on* $\mathbb{R}^d$ *and let* $A \in C^\infty$. *Then*

(I) $\int \alpha_{\theta x}(A) dE(x) = \int dE(x) \alpha_{\theta x}(A)$ (II) $\left(\int \alpha_{\theta x}(A) dE(x) \right)^* \subset \int \alpha_{\theta x}(A^*) dE(x)$

Moreover, let us introduce the deformed product, also known as the Rieffel product Rieffel (1993) by using warped convolutions. The two deformations are interrelated since warped convolutions supply isometric representations of Rieffel's strict deformations of C^*-dynamical systems with actions of $\mathbb{R}^d$.

Lemma 2. *Let* θ *be a real skew-symmetric matrix on* $\mathbb{R}^d$ *and let* $A, B \in C^\infty$. *Then*

$$A_\theta B_\theta \Phi = (A \times_\theta B)_\theta \Phi, \qquad \Phi \in \mathcal{D}.$$

where $\times_\theta$ *is known as the Rieffel product on* C^∞ *and is given by,*

$$(A \times_\theta B)\Phi = (2\pi)^{-d} \lim_{\varepsilon \to 0} \iint dx\, dy\, \chi(\varepsilon x, \varepsilon y)\, e^{-ixy}\, \alpha_{\theta x}(A) \alpha_y(B) \Phi. \tag{2}$$

The next proposition gives the transformation property of the warped convolution of an operator under the adjoint action of a unitary or anti-unitary operator on H. This is relevant since in Section 2 we examine the transformation properties of deformed operators under Poincaré transformations.

Proposition 3. *Let* W *be a unitary or anti-unitary operator on* H *such that* $WU(x)W^{-1} = U(Mx)$, $x \in \mathbb{R}^d$, *for some invertible matrix* M. *Then, for* $A \in C^\infty$,

$$WA_\theta W^{-1} = (WAW^{-1})_{\sigma M \theta M^T},$$

where M^T *is the transpose of* M *w.r.t the chosen bilinear form,* $\sigma = 1$ *if* W *is unitary and* $\sigma = -1$ *if* W *is anti-unitary.*

By using the former proposition and the homomorphism given in Grosse and Lechner (2007) we relate skew symmetric matrices θ to wedges $\mathcal{W}$. This in particular means that to each deformed operator with deformation matrix θ there is a corresponding wedge $\mathcal{W}$.

Most crucial to proving that the deformed fields satisfy a weakened locality known as wedge locality is the following proposition.

Proposition 4. *Let* $A, B \in C^\infty$ *be operators such that* $[\alpha_{\theta x}(A), \alpha_{-\theta y}(B)] = 0$ *for all* $x, y \in sp U$. *Then*

$$[A_\theta, B_{-\theta}] = 0.$$

In the next section we adopt Formula (1) to define warped convolutions with an unbounded operator X that is unitary equivalent to the momentum operator. Since we deform unbounded operators we are obliged to prove that the deformation formula, given as an oscillatory integral, is well-defined. This is the subject of the next section.

III | DEFORMING THE SCALAR QUANTUM FIELD

In this Section, we investigate the effect of deformation directly on a free scalar field. The unitary group used for deformation, is given by the infinitesimal generator X that is unitary equivalent to the momentum operator. Due to the unitary equivalence, the vector operator X is an essential self-adjoint operator on a dense domain and therefore defines a strongly continuous unitary group that we denote by $V(b) := e^{ibX}$. Furthermore, by using this Abelian group, an adjoint action can be defined and used for deformation in the framework of warped convolutions, Buchholz et al. (2011); Buchholz and Summers (n.d.).

Definition Let the one-particle Hilbert space be given as $H_1 := L^2(d^n\mu(\mathbf{p}),\mathbb{R}^n) = \{f : \int d^n\mu(\mathbf{p})|f(\mathbf{p})|^2 < \infty, d^n\mu(\mathbf{p}) := (2\omega_{\mathbf{p}})^{-1}\, d^n\mathbf{p}, (\omega_{\mathbf{p}},\mathbf{p}) \in H_m^+ := \{p \in \mathbb{R}^d | p^2 = m^2, p_0 > 0\}\}$ for $d-1 = n \geq 1$ and let $\Delta(P)$ be the dense domain of all functions from H_1 vanishing at infinity faster than any inverse polynomial in p^k given as follows, (Swieca & Voelkel, 1973, Equation III.24)

$$\Delta(P) = \{f \in H_1 : |\left(\mathbf{p}^2\right)^r f(\mathbf{p})| \leq c_r(f) < \infty; \quad r = 0,1,2,\ldots\}. \tag{3}$$

$\Delta(P)$ is contained in the domain of the essential self-adjoint momentum operators. The extended dense domain of the second quantized momentum operator P_μ is given by $\Delta_k(P) := \bigotimes_{i=1}^{k} \Delta(P)$ (for details concerning second-quantization see (Reed & Simon, 1975a, Theorem VIII.33) and (Reed & Simon, 1975a, Example 2)).

Definition Let the operator X_μ be defined by a unitary equivalence to the momentum operator as follows,

$$X_\mu = \Gamma(V^{-1})P_\mu\Gamma(V), \tag{4}$$

where the operator $\Gamma(V) := \bigoplus_{k=0}^{\infty} V^{\otimes k}$ is the second quantization of a unitary operator $V : H_1 \to H_1$ which may depend on several real parameters.

Proposition 5. *The operator X_μ defined by unitary equivalence (see Definition III) is an essentially self-adjoint operator on the dense domain*

$$\Delta_k(X) := \Gamma(V^{-1})\Delta_k(P), \tag{5}$$

commuting along its components, i.e.

$$[X_\mu, X_\nu] = 0.$$

Therefore, the following operator

$$V(p) = e^{ip_\mu X^\mu}, \tag{6}$$

is unitary and defines a strongly continuous group for all $p \in \mathbb{R}^d$.

Proof. By unitary equivalence essential self-adjointness of the operator X_μ follows. The density of the domain $\Delta_k(X)$ follows from the density of the unitary equivalent domain $\Delta_k(P)$ (see (Swieca & Voelkel, 1973, Lemma 2)). In order to show the commutation relations between the different components of the operator X_μ we use the unitary equivalence to the commuting momentum operators.

$$\begin{aligned}
[X_\mu, X_\nu] &= [\Gamma(V^{-1})P_\mu\Gamma(V), \Gamma(V^{-1})P_\nu\Gamma(V)] \\
&= \Gamma(V^{-1})[P_\mu, P_\nu]\Gamma(V) \\
&= 0,
\end{aligned}$$

where in the last line we used the fact that the momentum operator commutes along its components. Since X is an essential self-adjoint operator it follows that its closure is a self-adjoint operator and from ((Reed & Simon, 1975a, Theorem IIIX.7)) it follows that $V(p)$ defines a strongly continuous unitary group. □

Definition Let θ be a real skew-symmetric matrix w.r.t. the Lorentzian scalar-product on $\mathbb{R}^d$ and let $\chi \in S(\mathbb{R}^d \times \mathbb{R}^d)$ with $\chi(0,0) = 1$. Furthermore, let $\phi(f)$ be the massive free scalar field smeared out with functions $f \in S(\mathbb{R}^d)$. Then, the operator valued distribution $\phi(f)$ deformed with the operator X_μ (see Definition III), denoted as $\phi_{\theta,X}(f)$, is defined on vectors of the dense domain $\Delta_k(X)$ as follows

$$\begin{aligned}
\phi_{\theta,X}(f)\Psi_k :&= (2\pi)^{-d} \lim_{\varepsilon\to 0} \iint dx\,dy\, e^{-ixy} \chi(\varepsilon x, \varepsilon y) \beta_{\theta x}(\phi(f)) V(y)\Psi_k \\
&= (2\pi)^{-d} \lim_{\varepsilon\to 0} \iint dx\,dy\, e^{-ixy} \chi(\varepsilon x, \varepsilon y) \beta_{\theta x}\left(a(\overline{f^-}) + a^*(f^+)\right) V(y)\Psi_k \\
&=: \left(a_{\theta,X}(\overline{f^-}) + a^*_{\theta,X}(f^+)\right)\Psi_k.
\end{aligned} \tag{7}$$

The automorphism β is defined by the adjoint action of the unitary operator $V(y)$ and the test functions $f^\pm(\mathbf{p})$ in momentum space are defined as follows

$$f^\pm(\mathbf{p}) := \int dx\, f(x) e^{\pm ipx}, \qquad p = (\omega_{\mathbf{p}}, \mathbf{p}) \in H_m^+. \tag{8}$$

The integral (7) has to be understood as an integral in oscillatory sense, Rieffel (1993). The unboundedness of the operator X_μ questions the existence of the integral since we are dealing with unbounded operator valued distributions. To show that the integral (7) converges we use the unitary equivalence.

The following lemma proves the existence of a unitary transformation connecting the warped convolutions of a free scalar field using the momentum operator, and the warped convolutions of a free scalar field using the unitary equivalent operator X.

Lemma 6. *For $f \in S(\mathbb{R}^d)$ and $\Psi_k \in \Delta_k(X)$, a transformation exists that maps the field deformed with the momentum operator $\phi_{\theta,P}(f)$ to the field deformed with operator X, i.e. $\phi_{\theta,X}(f)$. This transformation is given as follows*

$$\phi_{\theta,X}(f)\Psi_k = \Gamma(V^{-1})\, \phi(Vf)_{\theta,P}\, \Gamma(V)\Psi_k. \tag{9}$$

Proof. By using the unitary equivalence given in Equation (4), the lemma is easily proven

$$\begin{aligned}
\phi_{\theta,X}(f)\Psi_k &= (2\pi)^{-d} \lim_{\varepsilon\to 0} \iint dx\,dy\, e^{-ixy} \chi(\varepsilon x, \varepsilon y)\, V(\theta x)\phi(f)V(-\theta x + y)\Psi_k \\
&= (2\pi)^{-d} \lim_{\varepsilon\to 0} \iint dx\,dy\, e^{-ixy} \chi(\varepsilon x, \varepsilon y)\, \Gamma(V^{-1})U(\theta x)\Gamma(V)\phi(f)\Gamma(V^{-1}) \\
&\qquad \times U(-\theta x + y)\Gamma(V)\Psi_k \\
&= \Gamma(V^{-1}) \left(\Gamma(V)\phi(f)\Gamma(V^{-1})\right)_{\theta,P} \Gamma(V)\Psi_k.
\end{aligned}$$

□

Lemma 7. *For $\Phi_k \in \Delta_k(X)$ the familiar bounds of the free field hold for the deformed field $\phi_{\theta,X}(f)$ and therefore the deformation with operator X_μ is well-defined.*

Proof. By using Lemma 6 one obtains the familiar bounds for a free scalar field. For $\Phi_k \in \Delta_k(X)$ there exists a $\Psi_k \in \Delta_k(P)$ such that the following holds

$$\begin{aligned}
\left\|\phi_{\theta,X}(f)\Phi_k\right\| &= \left\|\phi_{\theta,X}(f)\Gamma(V^{-1})\Psi_k\right\| \\
&= \left\|(\Gamma(V)\phi(f)\Gamma(V^{-1}))_{\theta,P}\Psi_k\right\| = \left\|(\phi(Vf))_{\theta,P}\Psi_k\right\| \\
&\leq \left\|(a(\overline{Vf^-}))_{\theta,P}\Psi_k\right\| + \left\|(a^*(Vf^+))_{\theta,P}\Psi_k\right\| \\
&\leq \left\|Vf^+\right\| \left\|(N+1)^{1/2}\Psi_k\right\| + \left\|Vf^-\right\| \left\|(N+1)^{1/2}\Psi_k\right\| \\
&= \left\|f^+\right\| \left\|(N+1)^{1/2}\Psi_k\right\| + \left\|f^-\right\| \left\|(N+1)^{1/2}\Psi_k\right\|.
\end{aligned}$$

where in the last lines we used the triangle inequality, the Cauchy-Schwarz inequality and the bounds given in (Grosse & Lechner, 2007).

The obtained bounds are exactly the bounds of the free scalar field. Thus by the bounds of the free field it follows that the field deformed with the operator X_μ is well-defined.

□

1 | Wightman Properties of the Deformed QF

It is important to note that due to the unitary equivalence we can show that the deformed field $\phi_{\theta,X}$ satisfies the Wightman properties with the exception of covariance and locality. This is the subject of the following proposition. We shall use the symbol H for Bosonic Fockspace and the symbol Ω to denote the vacuum.

Proposition 8. *Let θ be a real skew-symmetric matrix w.r.t. the Lorentzian scalar-product on $\mathbb{R}^d$ and $f \in S(\mathbb{R}^d)$.*

A) *The dense subspace $\mathcal{D}$ of vectors of finite particle number is contained in the domain $\mathcal{D}^{\theta,X} = \{\Psi \in H | \left\|\phi_{\theta,X}(f)\Psi\right\|^2 < \infty\}$ of any $\phi_{\theta,X}(f)$. Moreover, $\phi_{\theta,X}(f)\mathcal{D} \subset \mathcal{D}$ and $\phi_{\theta,X}(f)\Omega = \phi(f)\Omega$.*

B) *For scalar fields deformed via warped convolutions and $\Psi \in \mathcal{D}$,*

$$f \longmapsto \phi_{\theta,X}(f)\Psi$$

is a vector valued tempered distribution.

C) *For $\Psi \in \mathcal{D}$ and $\phi_{\theta,X}(f)$ the following holds*

$$\phi_{\theta,X}(f)^*\Psi = \phi_{\theta,X}(\overline{f})\Psi.$$

For real $f \in S(\mathbb{R}^d)$, the deformed field $\phi_{\theta,X}(f)$ is essentially self-adjoint on $\mathcal{D}$.

D) *The Reeh-Schlieder property holds: Given an open set of space-time $O \subset \mathbb{R}^d$, then*

$$\mathcal{D}_{\theta,X}(O) := span\{\phi_{\theta,X}(f_1)\dots\phi_{\theta,X}(f_k)\Omega : k \in \mathbb{N},$$

$$f_1 \dots f_k \in S(O)\} \textit{is dense in } H.$$

Proof. a) The fact that $\mathcal{D} \subset \mathcal{D}^{\theta,X}$, follows immediately from Lemma 7 , since the deformed scalar field satisfies the same bounds as a free field. The fact that the deformed field acting on the vacuum is the same as the free field acting on Ω, can be easily shown due to the property of the unitary operators $V(b)\Omega = \Omega$ (see (Reed & Simon, 1975b, Chapter X.7)).

b) By using Lemma 7 one can see that the right hand side depends continuously on the function f, hence the temperateness of $f \longmapsto \phi_{\theta,X}(f)\Psi$, $\Psi \in \mathcal{D}$ follows.

c) First, we prove hermiticity of the deformed field $\phi_{\theta,X}(f)$. This is done along the same lines as the proof of Lemma 1, demonstrating hermiticity of a deformed operator if the undeformed one is self-adjoint.

$$
\begin{aligned}
\phi_{\theta,X}(f)^*\Psi &= (2\pi)^{-d}\left(\lim_{\varepsilon\to 0}\iint dx\,dy\,e^{-ixy}\chi(\varepsilon x,\varepsilon y)\,\beta_{\theta x}(\phi(f))V(y)\right)^*\Psi \\
&= (2\pi)^{-d}\lim_{\varepsilon\to 0}\iint dx\,dy\,e^{-ixy}\,\overline{\chi(\varepsilon x,-\varepsilon y)}\,V(y)\beta_{\theta x}(\phi(f))^*\Psi \\
&= (2\pi)^{-d}\lim_{\varepsilon\to 0}\iint dx\,dy\,e^{-ixy}\,\overline{\chi(\varepsilon(x+\theta^{-1}y),-\varepsilon y)}\,\beta_{\theta x}(\phi(\overline{f}))V(y)\Psi \\
&= \phi_{\theta,X}(\overline{f})\Psi.
\end{aligned}
$$

In the last lines we performed a variable substitution $(y_\mu \to -y_\mu)$ and $\left(x_\mu \to x_\mu + (\theta^{-1}y)_\mu\right)$.

For real f we can prove the essential self-adjointness of the hermitian deformed field $\phi_{\theta,X}(f)$. The first step consists in showing that the deformed field has a dense set of analytic vectors. Next, by Nelson's analytic vector theorem, it follows that the deformed field $\phi_{\theta,X}(f)$ is essentially self-adjoint on this dense set of analytic vectors, (for similar proof see (Bratteli & Robinson, 1996, Chapter I, Proposition 5.2.3)).

For $\Psi_k \in H_k$ the estimates of the l-power of the deformed field $\phi_{\theta,X}(f)$, are given in the following

$$
\left\|\phi_{\theta,X}(f)^l\Psi_k\right\| \leq 2^{l/2}(k+l)^{1/2}(k+l-1)^{1/2}\cdots(k+1)^{1/2}\,\|f\|^l\,\|\Psi_k\|,
$$

where in the last lines we used Lemma 7 for the estimates of the deformed field. Finally, we can write the sum

$$
\sum_{l\geq 0}\frac{|t|^l}{l!}\left\|\phi(f)^l\Psi_k\right\| \leq \sum_{l\geq 0}\frac{(\sqrt{2}|t|)^l}{l!}\left(\frac{(k+l)!}{k!}\right)^{1/2}\|f\|^l\,\|\Psi_k\| < \infty
$$

for all $t \in \mathbb{C}$. It follows that each $\Psi \in \mathcal{D}$ is an analytic vector for the deformed field $\phi_{\theta,X}(f)$. Since the set $\mathcal{D}$ is dense in H, Nelson's analytic vector theorem implies that $\phi_{\theta,X}(f)$ is essentially self-adjoint on $\mathcal{D}$.

d) For the proof of the Reeh-Schlieder property we use the unitary equivalence given in Definition (III). First note that the spectral properties of the unitary operator $V(y)$, are the same as for the unitary operator $U(y)$ of translations. This leads to the application of the standard Reeh-Schlieder argument Streater and Wightman (1989) which states that that $\mathcal{D}_\theta(O)$ is dense in H if and only if $\mathcal{D}_\theta(\mathbb{R}^d)$ is dense in H. We choose the functions $f_1,\ldots,f_k \in S(\mathbb{R}^d)$ such that the Fourier transforms of the functions do not intersect the lower mass shell and therefore the domain $\mathcal{D}_\theta(\mathbb{R}^d)$ consists of the following vectors

$$
\begin{aligned}
\Gamma(V)\phi_{\theta,X}(f_1)\dots\phi_{\theta,X}(f_k)\Omega &= \Gamma(V)a^*_{\theta,X}(f_1^+)\dots a^*_{\theta,X}(f_k^+)\Omega \\
&= \Gamma(V)\Gamma(V^{-1})a^*_{\theta,P}(Vf_1^+)\dots a^*_{\theta,P}(Vf_k^+)\Gamma(V^{-1})\Omega \\
&= a^*_{\theta,P}(Vf_1^+)\dots a^*_{\theta,P}(Vf_k^+)\Omega \\
&= \sqrt{m!}P_m(S_m(Vf_1^+\otimes\dots\otimes Vf_k^+)),
\end{aligned}
$$

where P_k denotes the orthogonal projection from $H_1^{\otimes k}$ onto its totally symmetric subspace H_k, and $S_k \in B(H_1^{\otimes k})$ is the multiplication operator given as

$$
S_k(p_1,\dots,p_k) = \prod_{1\le l<j\le k} e^{ip_l\theta p_j}.
$$

Since the operator $\Gamma(V)$ is unitary, functions $Vf_k^+ \in S(\mathbb{R}^d)$ for $f_k^+ \in S(\mathbb{R}^d)$ will give rise to dense sets of functions in H_1. Following the same arguments as in Grosse and Lechner (2007) the density of $\mathcal{D}_\theta(\mathbb{R}^d)$ in H follows. Note that we proved the density for vectors $\Gamma(V)\phi_{\theta,X}(f_1)\dots\phi_{\theta,X}(f_k)\Omega$ and not for the vectors without the application of $\Gamma(V)$ as stated in the proposition. However, we use the unitarity of $\Gamma(V)$ to argue that vectors dense in H stay dense after the application of a unitary operator. □

2 | Wedge-Covariance and Wedge-Locality

The authors in Grosse and Lechner (2007) constructed a map $Q: W \mapsto Q(W)$ from a set $\mathcal{W}_0 := \mathcal{L}_+^\uparrow W_1$ of wedges, where $W_1 := \{x\in\mathbb{R}^d : x_1 > |x_0|\}$ to a set $Q_0 \subset \mathbb{R}^-_{d\times d}$ of skew-symmetric matrices. In the next step they considered the corresponding fields $\phi_W(x) := \phi(Q(W),x)$.

Hence, the correspondence is understood as a scalar field $\phi(Q(W),x)$ on a NC space-time, which can be equivalently realized as a field defined on the wedge. The homomorphism $Q: W \mapsto Q(W)$ is given by the following definitions.

Definition Let θ be a real skew-symmetric matrix on $\mathbb{R}^d$ then the map $\gamma_\Lambda(\theta)$ is defined as follows

$$
\gamma_\Lambda(\theta) := \begin{cases} \Lambda\theta\Lambda^T, & \Lambda\in\mathcal{L}^\uparrow, \\ -\Lambda\theta\Lambda^T, & \Lambda\in\mathcal{L}^\downarrow. \end{cases} \tag{10}
$$

Definition θ is called an admissible matrix if the realization of the homomorphism $Q(\Lambda W)$ defined by the map $\gamma_\Lambda(\theta)$ is a well defined mapping. This is the case iff θ has in d dimensions the following form

$$
\begin{pmatrix} 0 & \lambda & 0 & \cdots & 0 \\ \lambda & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \qquad \lambda \ge 0. \tag{11}
$$

For the physical most interesting case of 4 dimensions the skew-symmetric matrix θ has the more general form

$$\begin{pmatrix} 0 & \lambda & 0 & 0 \\ \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & \eta \\ 0 & 0 & -\eta & 0 \end{pmatrix}, \qquad \lambda \geq 0, \eta \in \mathbb{R}. \tag{12}$$

Note that the skew-symmetry is given w.r.t. the Minkowski metric.

By using the former definitions we give the following correspondence of the fields,

$$\phi_W(f) := \phi(Q(W), f) = \phi_{\theta,X}(f). \tag{13}$$

Next, we turn our attention to the covariance and locality of the defined fields. Wedge-covariance and -locality seems to be the appropriate locality on non-commutative space-times, Soloviev (2013). In the following we lay out the definitions of a wedge-covariant and a wedge-local field, (Grosse and Lechner (2007), Definition 3.2).

Definition Let $\phi = \{\phi_W : W \in \mathcal{W}_0\}$ denote the family of fields satisfying the domain and continuity assumptions of the Wightman axioms. Then, the field ϕ is defined to be a wedge-local quantum field if the following conditions are satisfied:

- **Covariance:** For any $W \in \mathcal{W}_0$ and $f \in S(\mathbb{R}^d)$ the following holds

  $$U(y,\Lambda)\phi_W(f)U(y,\Lambda)^{-1} = \phi_{\Lambda W}(f \circ (y,\Lambda)^{-1}), \qquad (y,\Lambda) \in \mathcal{P}_+^\uparrow,$$
  $$U(0,j)\phi_W(f)U(0,j)^{-1} = \phi_{jW}(\overline{f} \circ (0,j)^{-1}),$$

  where j represents the space-time reflections, i.e. $x^\mu \to -x^\mu$.

- **Wedge-locality:** Let $W, \tilde{W} \in \mathcal{W}_0$ and $f \in S(\mathbb{R}^2)$.

  If

  $$\overline{W + \text{supp}\, f} \subset (\tilde{W} + \text{supp}\, g)',$$

then

$$[\phi_W(f), \phi_{\tilde{W}}(g)]\Psi = 0, \quad \Psi \in \mathcal{D}.$$

The prime in the former definition denotes the causal complement. The last definition can be given in a simpler form due to the geometrical properties of the wedges. This is the subject of the following lemma, (Grosse and Lechner (2007), Lemma 3.3).

Lemma 9. *Let $\phi = \{\phi_W : W \in \mathcal{W}_0\}$ denote the family of fields satisfying the domain, continuity and covariance assumptions stated in Definition 2. Then ϕ is wedge-local if and only if*

$$[\phi_{W_1}(f), \phi_{-W_1}(g)]\Psi = 0, \quad \Psi \in \mathcal{D},$$

for all $f, g \in C_0^\infty(\mathbb{R}^d)$ with supp $f \subset W_1$ and supp $g \subset -W_1$.

So let us first investigate the wedge-covariance properties of our deformed fields. The result is given in the following proposition.

Proposition 10. *The deformed fields $\phi_{\theta,X}(f)$ transform under the adjoint action of the proper orthochronous Poincaré group as follows,*

$$U(x,\Lambda)\phi_{\theta,X}(f)U(x,\Lambda)^{-1} = \phi_{\theta,U(x,\Lambda)XU(x,\Lambda)^{-1}}(f\circ(x,\Lambda)^{-1}).$$

Let the operator X be covariant w.r.t. the proper orthochronous Lorentz group. Then, the field is wedge-covariant w.r.t. the proper orthochronous Lorentz group i.e.

$$U(0,\Lambda)\phi_{\theta,X}(f)U(0,\Lambda)^{-1} = \phi_{\gamma_\Lambda(\theta),X}(f\circ(0,\Lambda)^{-1}).$$

Moreover, if the operator X is covariant w.r.t. the proper orthochronous Poincaré group and the space-time reflections, then the field ϕ is a wedge-covariant field.

Proof.

$$\begin{aligned}
&U(x,\Lambda)\phi_{\theta,X}(f)U(x,\Lambda)^{-1}\\
&= (2\pi)^{-d}\lim_{\varepsilon\to 0}\iint dy\,du\,e^{-iyu}\chi(\varepsilon y,\varepsilon u)U(x,\Lambda)\beta_{\theta y}(\phi(f))V(u)\,U(x,\Lambda)^{-1}\\
&= (2\pi)^{-d}\lim_{\varepsilon\to 0}\iint dy\,du\,e^{-iyu}\chi(\varepsilon y,\varepsilon u)V_{\Lambda,x}(\theta y)\,\phi(f\circ(x,\Lambda)^{-1})\,V_{\Lambda,x}(-\theta y+u)\\
&= \phi_{\theta,U(\Lambda,x)XU(\Lambda,x)^{-1}}(f\circ(x,\Lambda)^{-1}),
\end{aligned}$$

where $V_{\Lambda,x}(y) := U(x,\Lambda)V(y)U(x,\Lambda)^{-1} = e^{iyU(x,\Lambda)XU(x,\Lambda)^{-1}}$. Now this expression is nothing else than the operator X used for deformation but unitarily transformed. The second and third part follow from Proposition 3, where in the case of space-time reflections one replaces the smearing function f with $\overline{f}$. □

Remark An operator that is translation invariant is not equivalent to the momentum operator, for example

$$X = U(\Lambda)PU(\Lambda)^{-1}, \qquad X = e^{iaD}Pe^{-iaD},$$

where D is the dilatation operator which is only essentially self-adjoint in the massless case, Wess (1959).

What information do we gain from the former proposition? It gives us the transformational behavior of a field defined on a wedge that can be associated with a excitation on a non-commutative space-time. Under the assumption of Lorentz covariance for the operator, it states that the field obtained by a Poincaré transformation associates to a transformed field generated by deformation with $U(x)XU(x)^{-1}$. The interpretation of the result is the following. Since the deformed fields generated by X are associated to a non-commutative space-time, fields generated by $U(x)XU(x)^{-1}$ correspond to fields on an equivalent but translated quantum space-time. Hence, we already are able to deduce from this result that generators of constant quantum space-times shall be translationally invariant. This will be further studied in Section V where we examine the isomorphism to non-commutative space-times.

Next we turn to the original proof of wedge-locality. It is usually done by showing that the functions used to smear the field are entire analytic and therefore they can be analytically continued to the complex upper half plane. The proof is done by introducing suitable coordinates given by,

$$m_\perp := (m^2+p_\perp)^{1/2}, \qquad p_\perp := (p_2,\dots,p_n), \qquad \vartheta := \frac{p_1}{m_\perp}.$$

In the new coordinates we have the following measure and on-shell momentum vector,

$$d^n\mu(\mathbf{p}) = d^{n-1}p_\perp d\vartheta, \qquad p(\vartheta) := \begin{pmatrix} m_\perp \cosh\vartheta \\ m_\perp \sinh\vartheta \\ p_\perp \end{pmatrix}$$

By using these new coordinates, the analyticity of the function and the analytic continuation one obtains for the smeared functions $f \in C_0^\infty(W_1)$ and $g \in C_0^\infty(-W_1)$, (see 22)

$$f^-(p_\perp, \vartheta + i\pi) = f^+(-p_\perp, \vartheta), \qquad g^-(p_\perp, \vartheta + i\pi) = g^+(-p_\perp, \vartheta). \tag{14}$$

Now for the proof of wedge-locality we have to demand that the unitary transformed functions, i.e. $Vf^-(p_\perp, \vartheta)$ and $Vg^+(p_\perp, \vartheta)$ satisfy the demanded analyticity and analytical continuation properties. Note that this restrains the unitary operators used in the definition of the operator that is unitary equivalent to the momentum operator. By taking the former definition and lemma into account the following proposition concerning the deformed field $\phi_{\theta,X}$ follows.

Proposition 11. *Let the unitary transformation V leave the support for all $f, g \in C_0^\infty(\mathbb{R}^d)$ with supp $f \subset W_1$ and supp $g \subset -W_1$ covariant, i.e.*

$$supp\ Vf \subset W_1 \qquad supp\ Vg \subset -W_1.$$

Then, the family of fields $\phi = \{\phi_W : W \in \mathcal{W}_0\}$ defined by $\phi_W(f) := \phi(Q(W), f) = \phi_{\theta,X}(f) = \phi_{\theta,V^{-1}PV}(f)$ are wedge-local fields on the Bosonic Fockspace H^+.

Proof. For the proof we use Proposition 4, the unitary equivalence given in Lemma 6 and the proof that the free scalar field deformed with the momentum operator is wedge local, Grosse and Lechner (2007). To use Proposition 4, we have to show that the following commutator vanishes for $f \in C_0^\infty(W_1)$ and $g \in C_0^\infty(-W_1)$,

$$\begin{aligned}[\beta_{\theta x}(\phi(f)), \beta_{-\theta y}(\phi(g))] &= [\beta_{\theta x}(a(\overline{f^-})), \beta_{-\theta y}(a^*(g^+)] - [\beta_{-\theta y}(a(\overline{g^-})), \beta_{\theta x}(a^*(f^+)] \\ &= \Gamma(V^{-1})[\alpha_{\theta x}\big(\phi(Vf)\big), \alpha_{-\theta y}\big(\phi(Vg)\big)]\Gamma(V),\end{aligned}$$

where in the former lines all other terms are equal to zero and the unitary equivalence was used. Let us first take a look at the first expression of the commutator,

$$\begin{aligned}&[\alpha_{\theta x}\big(\Gamma(V)a(\overline{f^-})\Gamma(V^{-1})\big), \alpha_{-\theta y}\big(\Gamma(V)a^*(g^+)\Gamma(V^{-1})\big)] \\ &= \int d^n\mu(\mathbf{p}) \int d^n\mu(\mathbf{k}) (Vf^-)(\mathbf{p})(Vg^+)(\mathbf{k}) e^{-ip\theta x} e^{-ik\theta y} [a(\mathbf{p}), a^*(\mathbf{k})] \\ &= \int d^n\mu(\mathbf{p}) (Vf^-)(\mathbf{p})(Vg^+)(\mathbf{p}) e^{-ip\theta(x+y)} \\ &= \int d^{n-1}p_\perp d\vartheta\, (Vf^-)(p_\perp, \vartheta)(Vg^+)(p_\perp, \vartheta) e^{-ip(\vartheta)\theta(x+y)} \\ &= \int d^{n-1}p_\perp d\vartheta\, (Vf^+)(-p_\perp, \vartheta)(Vg^-)(-p_\perp, \vartheta) e^{-ip(\vartheta+i\pi)\theta(x+y)} \\ &= \int d^n\mu(\mathbf{p}) (Vf^+)(\mathbf{p})(Vg^-)(\mathbf{p}) e^{ip\theta(x+y)},\end{aligned}$$

where in the last lines we used the unitary equivalence (4), the boundedness and analyticity properties of the unitary transformed functions f, g (see (Grosse & Lechner, 2007, Proposition 3.4)) and we shifted the contour of the integral from $\mathbb{R}$ to $\mathbb{R} + i\pi$. Next, we look at the second expression of the commutator and obtain the following,

$$
\begin{aligned}
&[\alpha_{-\theta y}\big(\Gamma(V)a(\overline{g^-})\Gamma(V^{-1})\big),\alpha_{\theta x}\big(\Gamma(V)a^*(f^+)\Gamma(V^{-1})\big)]\\
&=\iint d^n\mu(\mathbf{p})d^n\mu(\mathbf{k})(Vf^+)(\mathbf{p})(Vg^-)(\mathbf{k})e^{ip\theta x}e^{ik\theta y}[a(\mathbf{k}),a^*(\mathbf{p})]\\
&=\int d^n\mu(\mathbf{p})(Vf^+)(\mathbf{p})(Vg^-)(\mathbf{p})e^{ip\theta(x+y)}.
\end{aligned}
$$

Since the second expression of the commutator $[\beta_{\theta x}(\phi(f)),\beta_{-\theta y}(\phi(g))]$ is equal to the first one with a sign difference, the commutator vanishes. Hence, the fields ϕ_W are wedge-local. □

Concerning the wedge-covariance we imposed a strong requirement on the choice of our unitary operators. In particular, we demanded the unitary transformation V to leave the support for all $f,g \in C_0^\infty(\mathbb{R}^d)$ with $\operatorname{supp} f \subset W_1$ and $\operatorname{supp} g \subset -W_1$ covariant. Are there any examples of such transformations? This question will be answered positively by introducing a few examples.

Example We first mention the Lorentz-transformation, i.e. $\Gamma(V)=U(\Lambda)$ with $\Lambda\in \mathrm{SO}(1,1)\times\mathrm{SO}(d-2)$.

Proof. For the right wedge, i.e. $x_1>|x_0|$ we have to verify the following inequality

$$(\Lambda x)_1 > |(\Lambda x)_0|.$$

This can be easily verified by using the property of the wedge and the explicit form of the Lorentz boost in $0-1$ direction,

$$
\begin{aligned}
-\gamma\beta x_0+\gamma x_1 &> |\gamma x_0-\gamma\beta x_1|\\
-\beta x_0+x_1 &> |\,x_0-\beta x_1|,
\end{aligned}
$$

since the Lorentz-factor $\gamma>0$, moreover

$$-\beta x_0+x_1 > -\beta x_0+|x_0|>0,$$

since the velocity coefficient $|\beta|<1$. Thus we obtain

$$
\begin{aligned}
(-\beta x_0+x_1)^2 &> (x_0-\beta x_1)^2\\
x_1^2(1-\beta^2) &> x_0^2(1-\beta^2)\\
x_1 &> |x_0|.
\end{aligned}
$$

The proof for the left wedge is analogous. □

This example is well known and intuitively easy to understand, since the group $\mathrm{SO}(1,1)\times\mathrm{SO}(d-2)\subset\mathcal{L}_+^\uparrow$ is the stabilizer group $\mathcal{L}_+^\uparrow(W_1,)\subset\mathcal{L}_+^\uparrow$ of W_1.

Example Second we mention the unitary operator of translations in the momentum space $\Gamma(V(\vec{k}))=e^{i\vec{k}\cdot\vec{X}}$, where $\vec{X}$ is the second-quantized Newton-Wigner-Pryce operator. It was studied thoroughly in a QFT-context in Much (2013, 2015). In particular the operator acts on the particle operators as follows,

$$\Gamma(V(\vec{k}))a(\mathbf{p})\Gamma(V(\vec{k})^{-1})=a(\mathbf{p}-\mathbf{k}),\qquad \Gamma(V(\vec{k}))a^*(\mathbf{p})\Gamma(V(\vec{k})^{-1})=a^*(\mathbf{p}-\mathbf{k}).$$

Since the coordinate space remains invariant under such a transformation in the momentum space, the momentum-translation $V(\vec{k})$ is our second most prominent example.

Example Note that the translation operator $U(y) = e^{iy_\mu P^\mu}$ for $y \in W_1$ leaves the support of $f \in C_0^\infty(\mathbb{R}^d)$ covariant (see Buchholz et al. (2011)). By using the former examples we can take arbitrary arrangements of the operators $U(y)$, $\Gamma(V(\vec{k}))$ and $U(\Lambda)$ and thus obtain a whole class of wedge-local fields.

Example Although we intend to focus on the massless scalar field in a forthcoming work, we mention in this context the special conformal transformation. This operator leaves the wedge covariant and was intensively studied in Much (2012). The operator $\Gamma(V)$ that gives the unitary equivalence to the momentum operator is in the special conformal case the inversion operator constructed by Swieca and Voelkel (1973).

Example Another interesting unitary operator that should be mentioned in the massless case is given by the dilation operator, i.e. $\Gamma(V) = e^{ibD}$. It leaves the wedge covariant, since it represents merely a scale transformation and the operator $\Gamma(V^{-1})P\Gamma(V)$ transforms covariantly under Poincaré transformations.

The reader should be aware of the fact that wedge-covariance was not shown for the field $\phi_{\theta,X}$ although it is obligatory when proving wedge-locality. Nevertheless, by reducing the proof of wedge-locality for the field $\phi_{\theta,X}(f)$ to the field $\phi_{\theta,P}(Vf)$, we were able to circumvent this particular problem.

IV | SCATTERING

The next task of this work is to calculate the Scattering-matrix by using **tempered polarization free generators**, (Borchers et al., 2001; Schroer, 1997). In (Borchers et al., 2001) a framework was developed to calculate two-particle scattering of such a given theory, where the construction relies on the Haag-Ruelle scattering theory. In order to proceed let us briefly lay out the necessary definitions and properties.

Definition Let $W \in \mathcal{W}_0$ and $f \in S(\mathbb{R}^d)$. Then the following properties constitute a tempered polarization free generator $\phi_W(f)$,

A) $\phi_W(f)$ is a wedge-local field.

B) $\phi_W(f)$ and $\phi_W(f)^*$ are closed operators with Ω contained in their respective domains.

C) $\phi_W(f)\Omega$ and $\phi_W(f)^*\Omega$ are single particle states.

D) $\phi_W(f)$ is said to be **temperate** if there is a dense subspace $\mathcal{D}$ of its domain which is stable under translation, such that

$$x \mapsto \phi_W(f)U(x)\Psi, \qquad \forall \Psi \in \mathcal{D} \tag{15}$$

is strongly continuous and polynomially bounded in the norm for large x.

Lemma 12. *Let $W \in \mathcal{W}_0$ and $f \in S(\mathbb{R}^d)$ and let the unitary operator $\Gamma(V)$ be as demanded in Proposition 11. Then, the set of fields $\phi(Q(W), f) = \phi_{\theta,X}(f)$ constitute the properties of tempered polarization free generators.*

Proof. The first item in Definition IV is wedge-locality. This follows easily from the choice of the unitary operator $\Gamma(V)$ (see Proposition 11). Item b) holds since, $\phi_{\theta,X}(f)$ is a densely defined and symmetric operator, hence closeable. To see that the vacuum vector is contained in the domain see Proposition 8, a). It is straightforward to prove that the deformed fields generate single particle states and this property follows as well from Proposition 8, a).

Now let us turn our attention to the hardest part of this proof, the temperateness. Proving continuity for the expression $\phi_{\theta,X}(f)U(x)\Psi$ is equivalent to proving it for $\phi_{\theta,P}(Vf)\Gamma(V)U(x)\Psi$,

$$\begin{aligned}
&||\phi_{\theta,P}(Vf)\Gamma(V)U(x)\Psi - \phi_{\theta,P}(Vf)\Gamma(V)\Psi|| = ||\phi_{\theta,P}(Vf)\Gamma(V)(U(x)-1)\Psi||\\
&\leq \left\|Vf^{+}\right\| \left\|(N+1)^{1/2}\Gamma(V)(U(x)-1)\Psi\right\| + \left\|Vf^{-}\right\| \left\|(N+1)^{1/2}\Gamma(V)(U(x)-1)\Psi\right\|\\
&= \left\|f^{+}\right\| \left\|(N+1)^{1/2}(U(x)-1)\Psi\right\| + \left\|f^{-}\right\| \left\|(N+1)^{1/2}(U(x)-1)\Psi\right\| \xrightarrow{x\to 0} 0,
\end{aligned}$$

where in the last lines we used the fact that $\Psi \in \mathcal{D}$ and by applying unitary operators that do not change the particle number on vectors of finite particle number, we have $\Gamma(V)U(x)\Psi \in \mathcal{D}$ and hence we can use the bounds given in Lemma 7. Moreover, in the last expression we use the strong continuity of U for $\Psi \in \mathcal{D}$ and thus the scalar product and the limit can be interchanged.

Of course, the boundedness can be proven by following similar arguments as for the continuity. Nevertheless, a more elegant route is chosen, i.e.

$$\begin{aligned}
&||\phi_{\theta,X}(f)U(x)\Psi|| = (2\pi)^{-d}||\lim_{\varepsilon\to 0}\iint dy\,du\,e^{-iyu}\chi(\varepsilon y,\varepsilon u)\beta_{\theta y}(\phi(f))V(u)\,U(x)\Psi||\\
&= (2\pi)^{-d}||\lim_{\varepsilon\to 0}\iint dy\,du\,e^{-iyu}\chi(\varepsilon y,\varepsilon u)V_{-x}(\theta y)\,\phi(f\circ(-x))\,V_{-x}(-\theta y+u)\,\Psi||\\
&= ||\phi_{\theta,U(x)^{-1}XU(x)}(f\circ(x))\Psi||\\
&\leq \left\|V(f^{+}\circ(x))\right\| \left\|(N+1)^{1/2}\Psi\right\| + \left\|V(f^{-}\circ(x))\right\| \left\|(N+1)^{1/2}\Psi\right\|\\
&= \left\|f^{+}\right\| \left\|(N+1)^{1/2}\Psi\right\| + \left\|f^{-}\right\| \left\|(N+1)^{1/2}\Psi\right\|.
\end{aligned}$$

where $V_{-x}(y) = U(x)^{-1}V(y)U(x) = e^{iyU(x)^{-1}XU(x)}$. Now this expression is nothing else than the operator X used for deformation but unitarily transformed. Hence, we simply have another operator that is unitary equivalent to the momentum operator. By using the bounds in Lemma 7, boundedness follows. □

Next, let us define a function for $t \in \mathbb{R}$ and $f \in S(\mathbb{R}^d)$ by,

$$f_t(x) = (2\pi)^{-d/2}\int dp\,\widetilde{f}(p)\,e^{ipx}e^{i(p_0-\omega_{\mathbf{p}})t}.$$

The support properties of the functions f_t for asymptotic t are used in the subsequent discussion. To proceed, let us define the velocity support of f by,

$$\Xi(f) = \{(1,\mathbf{p}/\omega_{\mathbf{p}}) : p \in \operatorname{supp}\widetilde{f}\}.$$

It follows that the support of f_t is contained in $t\,\Xi(f)$. Furthermore, the partial ordering of the sets with reference to the wedge $\mathcal{W}_0$ have to be introduced.

Definition Let Ξ_a, $\Xi_b \subset \mathbb{R}^d$ be compact sets. Ξ_a is said to be the precursor of Ξ_b, $\Xi_a \prec \Xi_b$ in formula form, if $\Xi_a - \Xi_b$ is contained in $W \in \mathcal{W}_0$.

By using the former definitions and sophisticated techniques the authors were able to show that

$\phi_W(f_t)\phi_{W'}(g_t)\Omega$, converges to the incoming respectively outgoing two-particle states for $t \to \pm\infty$. For the test functions f, g with disjoint momentum supports in a small neighborhood of some point on the mass shell one obtains,

$$\lim_{t\to\infty} \phi_W(f_t)\phi_{W'}(g_t)\Omega = (\phi_W(f)\Omega \times \phi_{W'}(g)\Omega)_{out} \qquad \text{if} \qquad \Xi(g) \prec \Xi(f),$$
$$\lim_{t\to-\infty} \phi_W(f_t)\phi_{W'}(g_t)\Omega = (\phi_W(f)\Omega \times \phi_{W'}(g)\Omega)_{in} \qquad \text{if} \qquad \Xi(f) \prec \Xi(g),$$

where we used the standard notation for collision states. Our task is now to follow similar arguments made in (Grosse & Lechner, 2007) in order to calculate the amplitudes of a two-particle scattering. First note that the limits will depend on the wedge as well. Moreover, our model exhibits an independence of $t \in \mathbb{R}$ for the expression $\phi_W(f_t)\phi_{W'}(g_t)\Omega$. This in particular lies in the definition of $\phi_W(f_t)$, f^+, f_t^+ and the support properties of $\widetilde{f}$. The particular form of the scattering states are given in the following theorem.

Theorem 13. *Let $W \in \mathcal{W}_0$ and $f \in S(\mathbb{R}^d)$ and let the unitary operator V be as demanded in Proposition 11. Then the massive deformed field $\phi_{\theta,X}$ satisfies the properties of a tempered polarization free generator and the explicit form of two-particle scattering states are given for test functions $f, g \in S(\mathbb{R}^d)$ by*

$$\lim_{t\to\infty} \phi_W(f_t)\phi_{W'}(g_t)\Omega = \Gamma(V^{-1})\phi_{\theta,P}(Vf^+)\phi_{\theta,P}(Vg^+)\Omega \qquad \textit{if} \qquad \Xi(g) \prec \Xi(f),$$
$$\lim_{t\to-\infty} \phi_W(f_t)\phi_{W'}(g_t)\Omega = \Gamma(V^{-1})\phi_{\theta,P}(Vf^+)\phi_{\theta,P}(Vg^+)\Omega \qquad \textit{if} \qquad \Xi(f) \prec \Xi(g).$$

Proof. The property of tempered polarization is the main result of Lemma IV. The scattering states can be simply calculated by using the unitary equivalence given in Lemma 9.

□

V | ISOMORPHISM TO A NON-COMMUTATIVE SPACE-TIME

In (Grosse & Lechner, 2007) the deformed fields correspond to free fields defined on the representation space of the Moyal-Weyl plane $\mathcal{V}$. This correspondence is proven by defining a unitary operator which maps the Fock space H to the tensor product space $\mathcal{V} \otimes H$. Hence, the fields deformed with the momentum operator are on one hand wedge-covariant, wedge-local and nontrivial and on the other hand they correspond to free fields on a non-commutative space-time (NCST). Therefore, one question naturally arises in the context of this more general setting. To which NCST do fields, deformed with the unitary transformed operator, correspond to? In this section we partially answer this question by constructing a correspondence to a NCST.

Let $\mathcal{V}$ be the representation space of the *-algebra which is generated by the self-adjoint operators $\hat{x}$ that fulfill the commutator relation

$$[\hat{x}_\mu, \hat{x}_\nu] = -2i\theta_{\mu\nu}, \tag{16}$$

where θ is the center of the algebra. An isomorphism exists between the $*$-algebras of fields deformed with the momentum operator and the $*$-algebra of the free fields on non-commutative Minkowski space $\mathcal{V}$, (Grosse & Lechner, 2007). This equivalence is given by the following unitary operator

$V_{P,\xi} = \bigoplus_{n=0}^{\infty} V_{P,\xi}^{(n)} : H \to \mathcal{V} \otimes H$, with $\xi \in \mathcal{V}$ and $||\xi||_{\mathcal{V}} = 1$,

$$\left(V_{P,\xi}^{(n)} \Psi_n\right)(\mathbf{p}_1, \ldots, \mathbf{p}_n) = \Psi_n(\mathbf{p}_1, \ldots, \mathbf{p}_n) \cdot e^{i\sum_{k=1}^{n} p_k \hat{x}} \xi, \qquad \Psi_n \in H_n. \tag{17}$$

Hence, the following equations hold in a distributional sense

$$a_{\otimes,P}(\mathbf{p}) := e^{-ip\hat{x}} \otimes a(\mathbf{p}) = V_{P,\xi} a_{\theta,P}(\mathbf{p}) V_{P,\xi}^{*}, \tag{18}$$

where an analogous relation holds for the creation operator. Moreover, it follows from $V_{\theta,\xi}\Omega = \xi \otimes \Omega$ that the n-point functions of $\phi_{\otimes,P}$, i.e. the free fields on non-commutative Minkowski space, coincide with the those of the deformed field $\phi_{\theta,P}$,

$$\langle(\xi \otimes \Omega), \phi_{\otimes,P}(f_1) \ldots \phi_{\otimes,P}(f_n)(\xi \otimes \Omega)\rangle = \langle \Omega, \phi_{\theta,P}(f_1) \ldots \phi_{\theta,P}(f_n)\Omega\rangle. \tag{19}$$

Now since we deform with operators other than the momentum operator, we should obtain an isomorphism describing the equivalence of the deformed fields with fields living on different non-commutative space-times. These space-times correspond in a certain manner to the Moyal-Weyl since we deform with operators that are unitary equivalent to the momentum operator, that in turn generates the Moyal-Weyl spacetime. One path leading to the newly generated non-commutative space-time is by using the twist deformation (see Akofor, Balachandran, Jo, and Joseph (2007); Chaichian and Vernov (2011); Grosse and Wulkenhaar (2003); Soloviev (2013); Tureanu (2006); Zahn (2006) and references therein). In particular one could calculate the NC space-time by using the twisted commutator between the coordinates as already done for the special conformal operator in (Much, 2012). Next, we examine the equivalence of our deformed fields with the twist deformation approach. In this context the next lemma gives a unitary operator mapping the deformed fields $\phi_{\theta,X}$ to fields on a non-commutative space.

Proposition 14. *Let the unitary operator* $\widetilde{V}_{X,\xi} = \bigoplus_{n=0}^{\infty} \widetilde{V}_{X,\xi}^{(n)} : H \to \mathcal{V} \otimes H$, *with* $\xi \in \mathcal{V}$ *and* $||\xi||_{\mathcal{V}} = 1$, *be given by unitarily equivalence to* $V_{P,\xi}$ *as follows,*

$$\widetilde{V}_{X,\xi} = (\mathbb{1}_{\mathcal{V}} \otimes \Gamma(V^{-1}))V_{P,\xi}\Gamma(V). \tag{20}$$

Then $\widetilde{V}_{X,\xi}$ *is an isomorphism of the* $*$*- algebras generated by the deformed fields* $\phi_{\theta,X}(f)$ *to unitary equivalent* $*$*- algebras of the unitary transformed fields on the Moyal-Weyl space.*

Proof. Prior to the proof let us give the following expression,

$$V_{P,\xi}\, \phi_{\theta,P}(Vf)\, V_{P,\xi}^{*} = \phi_{\otimes,P}(Vf),$$

where this relation can be easily seen by the virtue of Equation (17). In the next step we calculate the adjoint action of $\widetilde{V}_{X,\xi}$ on the the deformed fields $\phi_{\theta,X}(f)$.

$$(\mathbb{1}_{\mathcal{V}} \otimes V^{-1})V_{P,\xi} \underbrace{V\left(\phi_{\theta,X}(f)\right)V^{-1}}_{\phi_{\theta,P}(Vf)} V_{P,\xi}^{*}(\mathbb{1}_{\mathcal{V}} \otimes V) = \qquad (\mathbb{1}_{\mathcal{V}} \otimes V^{-1})\phi_{\otimes,P}(Vf)(\mathbb{1}_{\mathcal{V}} \otimes V)$$

The equivalence can also be proven on the level of the n-point functions. Hence, we have shown that the deformed fields $\phi_{\theta,X}(f)$ are unitarily equivalent to the transformed fields that live on the Moyal-Weyl space, i.e. $\phi_{\otimes,P}(Vf)$. Note that the former equations hold in the sense of distributions. □

The following notation is self explanatory,

$$\phi_{\otimes,X}(f) := (\mathbb{1}_{\mathcal{V}} \otimes V^{-1})\phi_{\otimes,P}(Vf)(\mathbb{1}_{\mathcal{V}} \otimes V), \tag{21}$$

since we obtained this operator by an isomorphism from the deformed fields $\phi_{\theta,X}(f)$ to the tensor product space $\mathcal{V} \otimes H$. Next, we investigate in which sense our "twisted" fields $\phi_{\otimes,X}(f)$ fit into the framework of twisted deformation. In this deformation, the point-wise product of two scalar fields is replaced by the so called twist product. Let us give a precise mathematical definition of the former statement. Let $\mu : S(\mathbb{R}^d) \otimes S(\mathbb{R}^d) \to S(\mathbb{R}^d)$ denote the point-wise product of Schwartz functions. Then, the twisted product denoted by $\mu_{\theta,P}$ can be defined as $\mu_{\theta,P} = \mu \circ \mathcal{F}_{\theta,P}$ where

$$\mathcal{F}_{\theta,P} = e^{-i\theta_{\mu\nu}P^{\mu}\otimes P^{\nu}}. \tag{22}$$

In a remarkable paper (Zahn, 2006), the author gave a rigorous meaning to the twist product of two scalar fields by going to momentum space. Moreover, it was shown that there exists an equivalence between the product of twist deformed scalar fields and scalar fields introduced on the tensor product space $\mathcal{V} \otimes H$ given in (Doplicher, Fredenhagen, & Roberts, 1995). In particular the formula was given by

$$\phi_{\otimes,P}(f_1)\phi_{\otimes,P}(f_2) = \phi^2_{\otimes,P}(\mu \circ \mathcal{F}_{\theta,P}(f_1 \otimes f_2)), \tag{23}$$

where the following notation was introduced,

$$\phi^n_{\otimes,P}(f) = \int \prod_{i=1}^{n} dk_i \left(e^{i(k_1+\cdots+k_n)\hat{x}} \otimes \hat{f}(k_1,\ldots,k_n) \prod_{i=1}^{n} \check{\phi}(k_i) \right).$$

Remark The notation introduced in the context of twist-deformation is written off-shell. The reason therein lies in the extension of the twisted-QFT to scattering. However, in this work we shall proceed by going on-shell, i.e. $\check{\phi}(k) = \delta(k^2 - m^2)\tilde{\phi}(k)$.

By using the former notations and products of the twisted fields we are able to give the following lemma.

Lemma 15. *The product of two twisted fields $\phi_{\otimes,X}(f_1)\phi_{\otimes,X}(f_2)$ is given by unitary equivalence to the product of two Moyal-Weyl twisted fields as follows,*

$$\phi_{\otimes,X}(f_1)\phi_{\otimes,X}(f_2) = (\mathbb{1}_{\mathcal{V}} \otimes \Gamma(V^{-1}))\phi^2_{\otimes,P}(\mu \circ \mathcal{F}_{\theta,P}(Vf_1 \otimes Vf_2))(\mathbb{1}_{\mathcal{V}} \otimes \Gamma(V)).$$

Moreover, the product of n-twisted fields $\phi_{\otimes,X}(f_1),\ldots,\phi_{\otimes,X}(f_n)$ is given by unitary equivalence as

$$\phi_{\otimes,X}(f_1)\cdots\phi_{\otimes,X}(f_n) = (\mathbb{1}_{\mathcal{V}} \otimes \Gamma(V^{-1}))\phi^n_{\otimes,P}(\mu \circ \mathcal{F}_{\theta,P}(Vf_1 \otimes \cdots \otimes Vf_n))(\mathbb{1}_{\mathcal{V}} \otimes \Gamma(V)).$$

Proof. The products simply follow by the virtue of Equation (21) and by the fact that the operator V is unitary. □

One question still remains unsettled. How far do the twisted fields $\phi_{\otimes,X}(f)$ correspond to the twisting deformation framework? In particular if the operators generating the twist (22) are unitary equivalent to the momentum operator, do the fields $\phi_{\otimes,X}(f)$ represent the correct twisted field according to the deformation chosen? Hence, we take the unitarily transformed twist operator and calculate the non-commutative space-time. Next we calculate the product of two such twisted fields. Does this product correspond to the formula given in Equation (23)? These questions can be partially answered and are investigated by looking at simple examples.

Example Let the unitary operator $\Gamma(V)$ be given by the Lorentz-transformation $U(\Lambda)$. Then the twisted field $\phi_{\otimes,X}(f)$ is given by

$$\begin{aligned}\phi_{\otimes,X}(f) &= (\mathbb{1}_{\mathcal{V}}\otimes V^{-1})\int d^n\mu(\mathbf{k})\left(e^{ik\hat{x}}\otimes f^{-}(\Lambda^T\mathbf{k})a(\mathbf{k})+h.c.\right)(\mathbb{1}_{\mathcal{V}}\otimes V)\\ &= \int d^n\mu(\mathbf{k})\left(e^{ik\hat{x}}\otimes f^{-}(\Lambda^T\mathbf{k})a(\Lambda^T\mathbf{k})+h.c.\right)\\ &= \int d^n\mu(\mathbf{k})\left(e^{ik(\Lambda^T\hat{x})}\otimes f^{-}(\mathbf{k})a(\mathbf{k})+h.c.\right),\end{aligned}$$

where in the last lines we used the explicit result of the adjoint action of $U(\Lambda)$ on the particle operators, the representation of the field in $\mathcal{V}\otimes H$ (see Equation (18)) and the Lorentz-invariance of the measure. To compare this result with the twisted field obtained by deforming with the unitarily transformed twist, we have to rewrite the coordinate operators. It can be easily done in this simple case, since the new coordinate operators are simply $\hat{x}' := \Lambda^T\hat{x}$ with the following commutation relations,

$$[\hat{x}'_\mu,\hat{x}'_\nu] = -2i\left(\Lambda^T\theta\Lambda\right)_{\mu\nu}.$$

These are on the other hand the expected commutator relations of the coordinate operators that correspond to a NC space-time generated by the Lorentz-transformed twist. It is as well clear that a translation of the deformed field will not be noticed on since the operators $U(x)U(\Lambda)^{-1}PU(\Lambda)U(x)^{-1}$ and $U(\Lambda)^{-1}PU(\Lambda)$ generate the same **constant** NC space-time.

In the next example we choose to work in the massless case, i.e. with the Lorentz-invariant measure $d^n\mu(\mathbf{p}) := d^n\mathbf{p}(2|\mathbf{p}|)^{-1}$.

Example In the case of massless scalar fields we have an essentially self-adjoint operator D and therefore a strongly continuous one parameter group $\Gamma(V) = e^{ibD}$. Then the twisted field $\phi_{\otimes,X}(f)$ is given by

$$\phi_{\otimes,X}(f) = \int d^n\mu(\mathbf{k})\left(e^{ik(e^{-b}\hat{x})}\otimes f^{-}(\mathbf{k})a(\mathbf{k})+h.c.\right).$$

As before, we compare the result with the twisted field obtained by deforming with the unitarily transformed twist. Hence, we rewrite the coordinate operators as before, i.e. $\hat{x}' := e^{-b}\hat{x}$ with the following commutation relations,

$$[\hat{x}'_\mu,\hat{x}'_\nu] = -2ie^{-2b}\theta_{\mu\nu}.$$

These are on the other hand the expected commutator relations of the coordinate operators that correspond to a NC space-time generated by the scale-transformed twist. It is again clear that a translation of the deformed field will not be noticed since the operators $\Gamma(V^{-1})P\Gamma(V)$ generate a constant NC space-time. Since the NC space-time is constant it does not need to be translated with regards to the Poincaré transformed fields.

However, not all unitary transformations are as simple as the Lorentz-transformations or dilatations and thus it remains unclear to what extent the field $\phi_{\otimes,X}(f)$ corresponds to the respective twist deformation. This question shall be attacked more viciously in the context of algebraic QFT by the deformation of massless fields.

VI | CONCLUSION AND OUTLOOK

In this paper we established the existence of a broad class of deformations that result in wedge-locality and non-trivial two-particle scattering. Moreover, a new light was shed on the Poincaré transformational behavior of the deformed fields that correspond to fields living on a NC space-time. In fact it is required to incooperate the transformation into the quantized space-time.

A connection between the newly deformed fields and the scalar fields obtained by twist deformation was established as well. The connection is given by an explicit isomorphism. However, it is not clear if our fields defined on the tensor product space $\mathcal{V} \otimes H$ correspond to fields obtained by a twist deformation, other than the constant cases, i.e. $\Gamma(V) = U(\Lambda)$ or $\Gamma(V) = e^{ibD}$. To prove such an isomorphism we can extend the operator $\widetilde{V}_{X,\xi}$ to $(\tilde{U} \otimes \Gamma(V^{-1}))V_{P,\xi}\Gamma(V)$ and construct an explicit operator $\tilde{U}$. This will have to be studied with specific cases in order to be able to achieve a generalization for arbitrary V. Therefore, one should examine this isomorphism more thoroughly in the case of the massless field. The reason therein lies in the multiplicity of well studied unitary operators in the massless case that leave the wedge covariant. In particular, the conformal group provides a huge class of unitary operators for the investigation of the isomorphism to NC space-times.

The deformation was achieved with operators that are unitarily equivalent to the momentum operator. This may seem to be a restriction on the operators used for deformation. However, this is not very restrictive since by using the spectral theorem (Taylor, 2012, Chapter 8, Corollary 1.6) every set of commuting self-adjoint operators can be represented by a unitary equivalence to the momentum operator.

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