

Todos los caminos llevan a la condición de cuantización de Dirac

All roads lead to the Dirac quantization condition

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The existence of magnetic monopoles is a sufficient argument to explain the quantization of electric charge, an argument that was presented by Dirac. Regardless of the status of any search for magnetic monopoles, the formal description of the quantum mechanics of a charged particle in the field of a magnetic monopole is very rich and has increased our understanding of the mathematical structures underlying this description, as well as of its physical implications. In this short review, we present four different arguments all leading to the Dirac quantization condition, emphasizing their geometrical and topological aspects.

La existencia de monopolos magnéticos es un argumento suficiente para explicar la cuantización de la carga eléctrica, un argumento que fue presentado por Dirac. Más allá del estado de la búsqueda de monopolos magnéticos, la descripción formal de la mecánica cuántica de una partícula cargada en el campo de un monopolo magnético tiene mucha riqueza y ha aumentado nuestro entendimiento de las estructuras matemáticas que subyacen a esta descripción, así como de sus implicaciones físicas. En esta corta revisión, presentamos cuatro diferentes argumentaciones que llevan a la condición de cuantización de Dirac, enfatizando sus aspectos geométricos y topológicos.

PALABRAS CLAVES

monopolos magnéticos, teorías de norma, mecánica cuántica
no-asociativa, cohomología

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I | INTRODUCTION

It was discovered by Dirac (Dirac, 1931) that the existence of a single magnetic monopole somewhere in the Universe would be a sufficient condition for the discreteness of electric charge. We have an established fact of nature that the electric charge of charged objects comes in discrete units, with no deeper understanding of this fact to this day. In spite of a fervent and decades long experimental and observational search for magnetic monopoles, none has ever been seen (see (Rajantie, 2016), for

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instance).

Even if a magnetic monopole is never found, reworking the argument by Dirac is fruitful for several reasons. To begin with, it shows that a very elegant theoretical speculation may explain a very fundamental fact, and inspire attempts of quantization procedures of general physical systems (Nettel et al., 2009) or of coupling constants in field theory (Álvarez, 1985). But in addition to that, after Dirac's original presentation, a number of different argumentations have appeared that lead to the same quantization condition. These other paths employ mathematical tools which, besides being very powerful, are common to the whole of modern theoretical physics: differential geometry and topology. In this article, we simply review some of the existing presentations of the Dirac quantization condition. Through these, we explain in an informal but hopefully intuitive way the mathematical tools that make them possible. All presentations focus on different aspects of the same physical situation, that of the quantum description of a charged particle under the influence of a monopole's field. We first present in Section ?? the classical argument by Dirac, with the existence of a vector potential with string singularities. Then in Section III we briefly introduce the necessary background in cohomology and fibre bundles in order to present in Section IV the formulation of Wu & Yang in terms of non-singular local vector potentials with the quantization condition as a consistency condition for the existence of the underlying fibre bundle and in Section V the formulation of Álvarez where the quantization condition appears as a consistency condition for the freedom to choose any atlas on configuration space, later expressed as a non-trivial cohomology. Finally, in Section VI we present the formulation of Jackiw in which the quantization condition restores associativity of quantum operators.

II | DIRAC QUANTIZATION CONDITION

Let us reproduce the quantization condition in the spirit first proposed by Dirac. Let us assume that magnetic monopoles exist and consider that one with magnetic charge g located at the origin of coordinates produces a magnetic field \mathbf{B} given by

$$\nabla \cdot \mathbf{B} = 4\pi g \delta(\mathbf{r}). \quad (1)$$

Now, consider a surface S enclosing a volume V containing the magnetic monopole. By using Stokes' theorem we have

$$\int_S \mathbf{B} \cdot d\mathbf{a} = \int_V \nabla \cdot \mathbf{B} dV = 4\pi g \quad (2)$$

and therefore the total magnetic flux through the surface S is $4\pi g$. So far, there is no difference with the case of an electric charge. Notice that the magnetic field

$$\mathbf{B}(\mathbf{x}) = g \frac{\hat{\mathbf{x}}}{x^2} \quad (3)$$

satisfies the previous equations. In the following we take $c = \hbar = 1$.

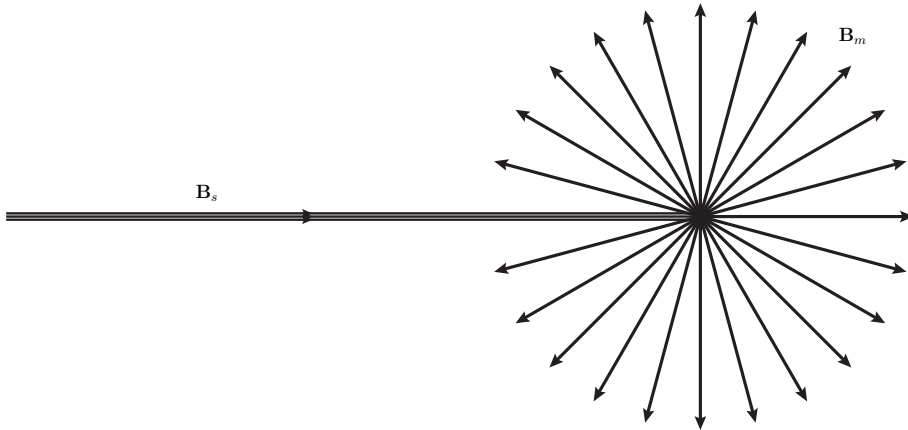


Figure 1: Magnetic field of an infinitely thin semi-infinite solenoid.

Dirac conceived the magnetic monopole as an infinitely thin semi-infinite solenoid (Coleman, 1982; Dirac, 1931). The resulting magnetic field can be split into two contributions, one corresponding to the usual solenoid field \mathbf{B}_s confined to its interior and another \mathbf{B}_m emanating from the end of the solenoid. The magnetic field at the end approaches that of a magnetic monopole (see Figure 1) in the limit in which the thickness of the solenoid decreases. This is fine as long as the field inside the solenoid, and with this the semi-infinite solenoid, becomes undetectable by any physical means. So, we could try to detect the solenoid by electron interference as in the Aharonov-Bohm effect (Aharonov and Bohm, 1959). There, we have a variation of a double slit experiment, with electrons emitted by a source O , passing by the two slits of a surface along trajectories L and R and finally interfering at a detector screen, but with the presence of the static magnetic field of the solenoid, as in Figure 2. As we know, the solenoid field is confined within its interior, which is specified as the region S .

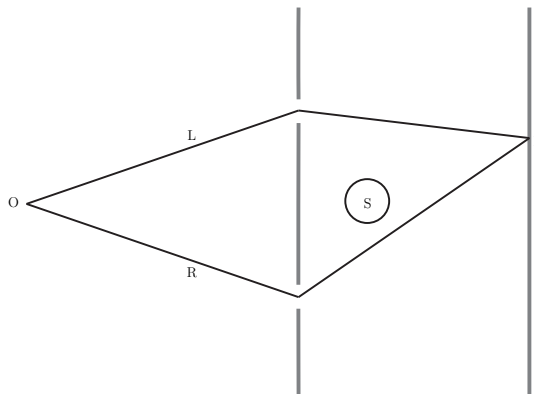


Figure 2: Setting for the Aharonov-Bohm effect.

The hamiltonian for an electron in the presence of the solenoid field \mathbf{B}_s is

$$H = \frac{1}{2m} (\mathbf{p} - e\mathbf{A}_s)^2, \quad (4)$$

where \mathbf{A}_s is the vector potential such that $\nabla \times \mathbf{A}_s = \mathbf{B}_s$ and e is the electron's charge. We can very easily verify that a solution $\psi(\mathbf{x}, t)$ of the corresponding Schroedinger equation

$$i \frac{\partial}{\partial t} \psi(\mathbf{x}, t) = H\psi(\mathbf{x}, t) \quad (5)$$

is of the form

$$\psi(\mathbf{x}, t) = \phi(\mathbf{x}, t) \exp \left(ie \int_{\mathbf{x}_0}^{\mathbf{x}} \mathbf{A}_s(\mathbf{x}') \cdot d\mathbf{x}' \right), \quad (6)$$

where $\phi(\mathbf{x}, t)$ is a solution to the Schroedinger equation for a free particle. Therefore, the presence of the magnetic field is reflected on the wavefunctions only by an addition of a phase. This phase is a line integral of \mathbf{A}_s performed along any path connecting \mathbf{x}_0 and \mathbf{x} in a region not including the solenoid, where $\nabla \times \mathbf{A}_s = 0$. In the Aharonov-Bohm setting, we have two slits through which the electron can pass, which we call L and R, so that the probability amplitude that the electron reaches point F is

$$\begin{aligned} & \phi_L(\mathbf{x}, t) \exp \left(ie \int_L \mathbf{A}_s(\mathbf{x}') \cdot d\mathbf{x}' \right) + \phi_R(\mathbf{x}, t) \exp \left(ie \int_R \mathbf{A}_s(\mathbf{x}') \cdot d\mathbf{x}' \right) \\ &= \exp \left(ie \int_L \mathbf{A}_s(\mathbf{x}') \cdot d\mathbf{x}' \right) \left[\phi_L(\mathbf{x}, t) + \exp \left(ie \int_R \mathbf{A}_s(\mathbf{x}') \cdot d\mathbf{x}' - ie \int_L \mathbf{A}_s(\mathbf{x}') \cdot d\mathbf{x}' \right) \phi_R(\mathbf{x}, t) \right]. \end{aligned} \quad (7)$$

The relative phase factor between the contributions from both slits involves the difference between the line integrals of \mathbf{A}_s along each path, which is actually a closed line integral. Therefore, the presence of the magnetic field contributes by a relative phase between the amplitudes for both paths given by

$$\oint_C \mathbf{A}_s(\mathbf{x}') \cdot d\mathbf{x}' = \int_R \mathbf{B}_s \cdot d\mathbf{a} = 4\pi g. \quad (8)$$

This is the well-known story that the electron's wavefunction picks up a phase due to the magnetic field even if travels through a region where this magnetic field is absent. However, the solenoid may turn out to be undetectable if this phase factor is trivial, that is, if

$$4\pi g e = 2\pi n, \quad (9)$$

with n some natural number. This is precisely the Dirac quantization condition. This way, the solenoid becomes undetectable and we are left only with the monopole field \mathbf{B}_m with total charge g .

III | SOME GEOMETRY AND TOPOLOGY

Let us begin by placing some electromagnetic objects in their proper geometric context. We assume some familiarity with the differential geometry of manifolds. A good introduction can be found in (Frankel, 2011; Nakahara, 2003), on which we base the following presentation. Typically we depict the vector potential and the magnetic field as vector fields in \mathbf{R}^3 . Although we will always pay homage to Faraday's genius, it might be useful to regard these as differential forms over an arbitrary manifold M . This identification between vectors and forms is an accident due to the dimensionality of 1- and 2-forms in three dimensional manifolds. More formally, the potential A is a differential 1-form field, while the magnetic field B is a 2-form field. These are instances of p -forms, which are totally antisymmetric

multilinear maps that are given p vectors and return real numbers.

1 | Cohomology

Let $\Omega^p(M)$ be the space of smooth p -forms on M , with $\Omega^0(M)$ the space of smooth functions on M . In the following, we will sometimes denote the order of the form by a superscript, as ω^p for a p -form. The exterior derivative d is an operator that satisfies the following properties:

1. $d : \Omega^p(M) \rightarrow \Omega^{p+1}(M)$, $\xi^q = d\omega^p \wedge \xi^q + (-1)^p \omega^p \wedge d\xi^q$,
2. let $\omega^p \in \Omega^p(M)$ and $\xi^q \in \Omega^q(M)$, then $d(\omega^p \wedge \xi^q) = d\omega^p \wedge \xi^q + (-1)^p \omega^p \wedge d\xi^q$, 3. $d^2 = 0$,

that is, d maps p -forms to $(p + 1)$ -forms, satisfies the (graded) Leibniz rule, like any derivative, and is nilpotent.

We recall Stokes' theorem. If ω^{p-1} is a $p - 1$ -form on M and V is a p -dimensional compact orientable submanifold in M with boundary ∂V , then

$$\int_{\partial V} \omega^{p-1} = \int_V d\omega^{p-1}. \tag{10}$$

If we define an "inner product" between p -forms ξ^p and p -surfaces W by $\langle W, \omega^p \rangle = \int_W \omega^p$, then Stokes' theorem can be expressed as $\langle \partial V, \omega^{p-1} \rangle = \langle V, d\omega^{p-1} \rangle$. This highlights a duality between the exterior derivative d acting on differential forms and the boundary operator ∂ acting on submanifolds.

We have sufficient ingredients to start introducing the concept of cohomology. Given a p -form ω^p , we say the ω is *closed* if $d\omega^p = 0$ and that ω^p is *exact* if $\omega^p = d\eta^{p-1}$, that is, if ω^p is the differential of a $(p - 1)$ -form η^{p-1} . Then the set of closed p -forms is called the cocycle group $Z^p(M)$, while the set the exact p -forms is called the coboundary group $B^p(M)$. Notice that, since the exterior derivative is nilpotent, then $B^p(M) \subset Z^p(M)$, but not necessarily the converse is true. Consider now two closed p -forms ω, ω' , differing by an exact form, $\omega^p = \omega'^p + d\eta^{p-1}$. They are said to be cohomologous. If we identify all cohomologous p -forms, we end up with the quotient space $H^p(M) = Z^p(M)/B^p(M)$, called the p -th cohomology group. Cohomology precisely measures the failure of closed forms to be exact.

This is just an example of a possible cohomology construction. We typically need to define *cochains*, in this case played by p -forms, and a nilpotent coboundary operator. The dual construction of a *homology* is defined in terms of some basic geometrical regions into which any submanifold can be decomposed called *chains* and a nilpotent boundary operator.

As an example, consider the form on \mathbb{R}^2 given by $\omega^1 = (x^2 + y^2)^{-1}(xdy - ydx)$. This form is not defined at the origin, so that the actual manifold on which the form is defined is $\mathbb{R}^2 - \{0\}$. One can easily show that ω^1 is closed. On the other hand, if we define the function $\eta = \arctan(y/x) = \theta$, then we can show that $d\eta = \omega^1$. However, this does not imply that ω^1 is exact. Note that θ is not a single-valued function, so that we have to introduce a branch cut, the positive x axis, for example. ω^1 is closed and exact on $\mathbb{R}^2 - \text{branch cut}$. One can actually move the branch cut somewhere else, since choosing the x axis was arbitrary, and make ω^1 locally exact over different pieces of $\mathbb{R}^2 - \{0\}$. This is actually guaranteed by Poincare's lemma, according to which around any point there is a sufficiently small neighborhood in which any closed form is locally exact. But if $\omega^1 = d\theta$ were globally exact, its integral along any closed curve would be zero due to Stokes theorem; however, its integral along the unit circle is 2π . In this case, one actually computes a non-trivial cohomology group $H^1(\mathbb{R}^2 - \{0\})$.

2 | Fibre bundles

The previous discussion on potentials and wavefunctions can be enlightened if we place these objects in their proper mathematical context, that we know as fibre bundle theory. An accesible introduction can be found in (Shapere and Wilczek, 2012). In this framework, the vector potential corresponds to a so called local connection and the wavefunction to a local section in a principal fibre bundle. It is very useful to review the subject of fibre bundles, since these appear in several very important physical theories, from the n -body problem (Littlejohn and Reinsch, 1997) to the gauge structure of Hilbert space in quantum mechanics (Bengtsson and Zyczkowski, 2017) to the Standard Model and gauge field theory in general (Peskin and Schroeder, 1995; Ryder, 1996), just to mention a few examples.

Consider a product space $M \times F$. In the world of fibre bundles, this is called a *trivial* fibre bundle. A fibre bundle is a generalization of product spaces, as a space that *locally* is a product space. A fibre bundle E contains a base manifold M and fibre homeomorphic to F such that in any neighborhood U around a point in M , the bundle looks like $U \times F$. A cylinder and a Möbius band are both locally of the form $S^1 \times [0, 1]$, but the latter is twisted. These two are examples of fiber bundles. Also, S^3 is locally of the form $S^2 \times S^1$, according to the Hopf fibration, but of course it is not globally factorized in that way. Points in the bundle are specified by the point p on the manifold and the point ξ on the fiber over p , as we illustrate in Figure 3. We define a projection $\pi : E \rightarrow M$ that maps all points ξ on the fibre over p to p . A section $\sigma : M \rightarrow E$ is an embedding of local patches of the base manifold into the bundle, with the requirement that $(\pi \circ \sigma)(p) = p$. Bundles in which a group G acts on the fibers are called G -bundles and when the fiber is G itself they are called *principal* G -bundles, typically denoted by P .

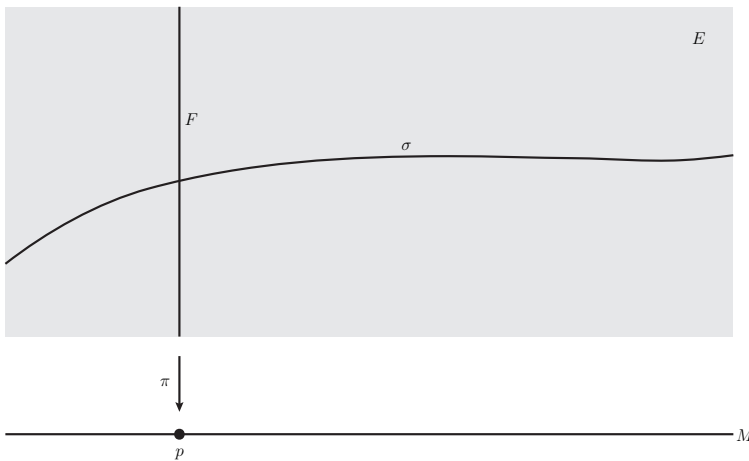


Figura 3: Fibre bundle structure.

At a point $u = (p, \xi)$ in the bundle E , we may try to decompose the local tangent space as some vertical subspace and some horizontal subspace, $T_u E = V_u \oplus H_u$. We define the vertical direction as that along the fibre. However, there is no natural identification between points over different fibers, so that there is no natural definition of the horizontal direction. For this to be defined, we have to implement an additional structure called the *connection* ω . This is a \mathfrak{g} -valued 1-form defined in the bundle that determines that a vector V in the bundle is horizontal if

$$\omega(V) = 0. \tag{11}$$

Here \mathfrak{g} is the Lie algebra associated to the group G . Being \mathfrak{g} -valued implies that it belongs to $\mathfrak{g} \otimes \Omega^1(M)$ and returns an element of \mathfrak{g} when given a vector. This notion of horizontality is tied to the particular connection chosen, that is, to a different connection corresponds different horizontal subspaces. Furthermore, by means of a section $\sigma : M \rightarrow E$, we may define a local connection 1-form A in the base manifold M as the pull-back $\sigma^* \omega$.

We may define the concept of holonomy in bundles where we have a structure group G . Given a curve γ in M , parameterized by $t \in [0, 1]$, there exist several ways to lift this curve in E . A connection allows us to define a horizontal lift as that in which the curve γ is lifted following the horizontal directions at each point in E . Consider that the horizontal lift of γ is given by $\tilde{\gamma}$. In terms of the local section σ , we may write that $\tilde{\gamma}(t) = \sigma(\gamma(t))g(t)$, with $g(t) \in G$. We may choose a section such that $\sigma(\gamma(0)) = \tilde{\gamma}(0)$. The condition of horizontal lift implies a condition on $g(t)$. If X is the vector tangent to γ , then $\tilde{X} = \tilde{\gamma}_* X$ is tangent to $\tilde{\gamma}$. Then we demand that \tilde{X} be horizontal, that is, $\omega(\tilde{X}) = 0$. One can show that this implies the differential equation

$$\frac{dg(t)}{dt} = -A(X)g(t), \quad g(0) = \text{identity}, \tag{12}$$

with solution

$$g(t) = \mathcal{P} \exp \left(- \int_{\gamma(0)}^{\gamma(t)} A \right), \tag{13}$$

the path-ordered exponential of A . When the curve γ is closed and we horizontally lift it in its entirety, the resulting total lift

$$g_\gamma = \mathcal{P} \exp \left(- \oint_\gamma A \right) \tag{14}$$

is called the *holonomy*. It is interesting that this group element might be non-trivial and therefore that the horizontal lift of a closed curve might be open. It might gain some intuition as to why the connections A or ω have to be \mathfrak{g} -valued. As we horizontally lift a curve, the connection determines the direction in which we move along the fibers, which is a direction in the Lie algebra \mathfrak{g} .

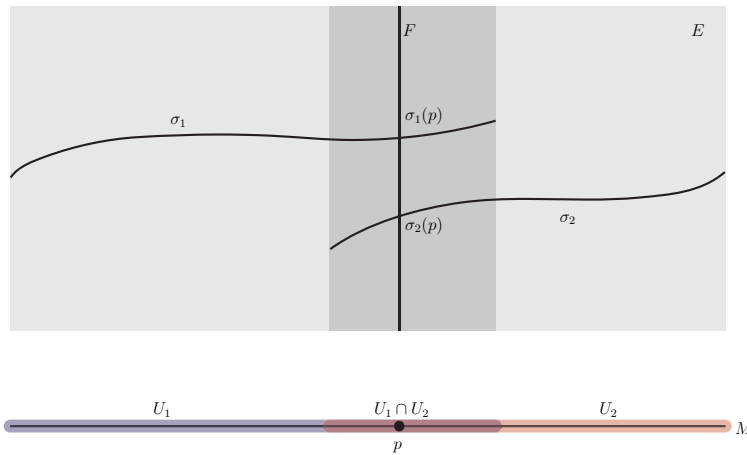


Figura 4: Gauge transformation.

Consider two patches U_1 and U_2 in M . We may choose a section σ_1 over U_1 and a section σ_2 over U_2 . At each point p in the overlap $U_1 \cap U_2$, there are two sections defined, which are related by a particular group element $g(p)$ in G , that is, $\sigma_2(p) = \sigma_1(p)g(p)$. This relation between sections is what we call in physics a gauge transformation. A gauge transformation implies a relation between the local connections A_1 and A_2 . Notice that the sections not only are defined over different patches in M , but consist of different regions in the bundle and therefore we are considering pull-backs of ω at different regions. A standard result is that

$$A_2 = g^{-1}A_1g + g^{-1}dg, \quad (15)$$

where d is the exterior derivative in M .

Finally, consider a principal fibre bundle P over M with fiber G . Consider a k -dimensional vector space V and a k -dimensional representation ρ of G that acts by the left on V . Given the action of G on $P \times V$ by $g \triangleright (u, v) = (ug, \rho(g^{-1})v)$, we define the equivalence relation in $P \times V$ such that (u, v) and $g \triangleright (u, v)$ are identified. Then the quotient space $E = P \times_{\rho} V = P \times V / G$ is the associated vector bundle with base space M , fiber V and structure group G . A vector bundle E may exist as an associated bundle to a principal bundle P or may exist in its own. Conversely, given a vector bundle E , it is possible to define a principal bundle P associated to it.

Consider a principal $U(1)$ -bundle P over a manifold M and its associated bundle $P \times_{\rho} \mathbb{C}$ with fiber \mathbb{C} . The latter is called a complex line bundle because the fiber is complex one dimensional. The representation ρ simply consists of phase factors. A section in the vector bundle over a patch M assigns a complex number to each point in M , that is, a complex function ψ on M . The group $U(1)$ acts on the section by $g \triangleright \psi = e^{i\phi} \psi$, simply by phase multiplication. Given two patches U_1 and U_2 in M , sections ψ_1, ψ_2 and local connection forms A_1, A_2 respectively defined on these patches, in a point p in the overlap $U_1 \cap U_2$ the sections are related by

$$\psi_2 = e^{i\phi} \psi_1, \quad (16)$$

while the local connections are related by

$$A_2 = e^{-i\phi} A_1 e^{i\phi} + e^{-i\phi} d e^{i\phi} = A_1 + i d\phi. \quad (17)$$

We will see that the vector potentials of electromagnetism are local connections in a complex line bundle.

IV | STRING SINGULARITY AND ITS TAMING

There is something very problematic about the previous presentation of the Dirac quantization condition that a careful reader must have already worried about. But also, to that careful reader, the previous discussion on cohomology must already be indicating a specific characterization of what the problem and its solution are. As we said, the monopole's magnetic field is specified by a 2-form B . Also, we have been considering a vector potential specified by a 1-form A , which we regarded as related to B by $B = dA$. But then it would happen that $dB = 0$, that is, there are no magnetic charges. The whole point of this review is to consider the quantum mechanics of charged particles under the field of magnetic charges. The reader is in her right to feel confused. It actually happens that the potential A associated to a magnetic monopole is necessarily singular along a string, as we will soon show. Later on, we will see how we can get rid off of these singularities, but only at the expense that the whole of space will have to be divided into patches, over which a *local* vector potential will be defined. We owe these ideas to the seminal work by Wu & Yang (Wu and Yang, 1975; Wu and Yang, 1976; Yang, 1977), laying the foundations for a broader application of the mathematical theory of fibre bundles in fundamental

physics.

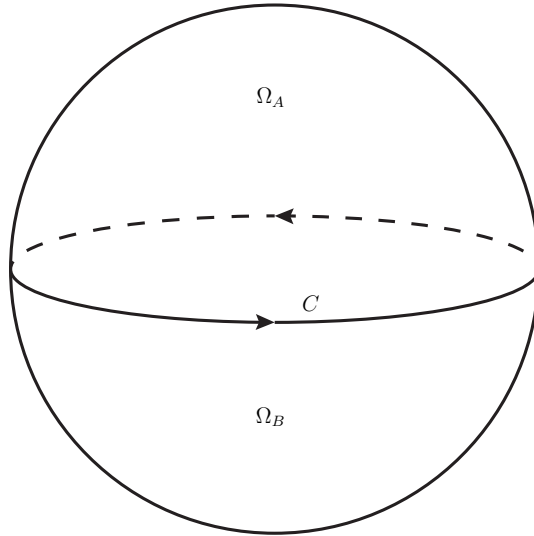


Figure 5: We divide the sphere in two caps Ω_A , Ω_B with common boundary C (with opposite orientation).

Consider a magnetic monopole of charge g in the center of a sphere of radius R . The cycle C divides the sphere in two caps Ω_A and Ω_B , as in figure 5. Then the total magnetic outward flux over the sphere is

$$4\pi g = \int_{S^2} B = \int_{\Omega_A} B + \int_{\Omega_B} B. \quad (18)$$

Now, assume that the associated vector potential A is free of singularities, so that we can apply Stokes' theorem to the surface integral over each cap, resulting in

$$\int_{\Omega_A} B + \int_{\Omega_B} B = \int_C A - \int_C A = 0, \quad (19)$$

which is a contradiction. Each sign in the preceding equation is due to the orientation of the cycle C consistent with the outward orientation of the normal vector on the sphere. This contradiction implies that the vector potential must be singular somewhere on the sphere. However, since the size of the sphere was irrelevant in the argument, the singularity must actually lie along an entire string. The magnetic field is free of singularities, though, and Wu & Yang have shown that the singularity in the vector potential is not physical, but rather associated to a singularity of a *global* description in the same sense that angular coordinates on the sphere are singular if we want them to cover the entire sphere.

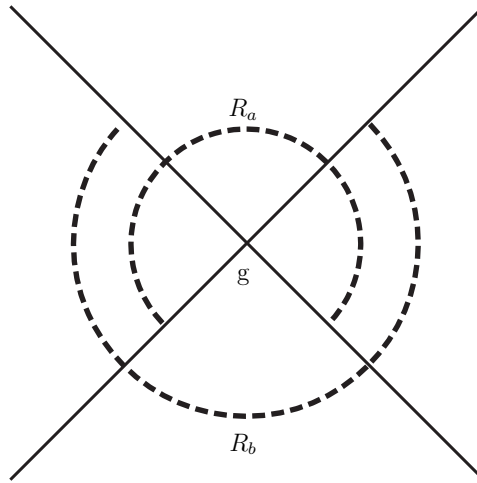


Figura 6: Patches in which we divide space and on which we divide local vector potentials, reproduced from (Yang, 1977).

We will thus eliminate the singularities by dividing space in two regions and defining a well-behaved vector potential in each of them. Figure 6 exhibits two cones in space with their vertices on the monopole position. R_a is the region above the lower cone, while R_b is the region below the upper cone. The union of both regions is the whole of space excluding the monopole. In each region, we define the potentials

$$\begin{aligned} R_a : \mathcal{A}_a &= iA_a = ig(1 - \cos\theta)d\phi, \\ R_b : \mathcal{A}_b &= iA_b = -ig(1 + \cos\theta)d\phi, \end{aligned} \tag{20}$$

where (r, θ, ϕ) are spherical coordinates in \mathbb{R}^3 and $d\phi$ is the dual to the tangent ∂_ϕ to the constant ϕ curves. We use objects \mathcal{A} and A because $u(1) \simeq i\mathbb{R}$, while the electromagnetic potentials have to be real. Each potential is well-defined over its own region and by taking $\mathcal{B} = d\mathcal{A} = iB$ we obtain the monopole magnetic field B . In the intersection R_{ab} of both regions, both potentials are defined and related in the usual manner for vector potentials, that is,

$$\mathcal{A}_a - \mathcal{A}_b = i(A_a - A_b) = 2igd\phi = d(2ig\phi), \tag{21}$$

as we can compute directly.

Let us formulate the description of the quantum behaviour of an electron under the magnetic field of a monopole. In the coordinate basis, the Schroedinger equations for the wave functions ψ_a and ψ_b when the electron is in regions R_a or R_b , respectively, are

$$\begin{aligned} -\frac{1}{2m} \left(\frac{\partial}{\partial x^\mu} - ie(A_a)_\mu \right)^2 \psi_a &= i \frac{\partial \psi_a}{\partial t}, \\ -\frac{1}{2m} \left(\frac{\partial}{\partial x^\mu} - ie(A_b)_\mu \right)^2 \psi_b &= i \frac{\partial \psi_b}{\partial t}. \end{aligned} \tag{22}$$

In the previous section we showed that the presence of the vector potential term in the Schroedinger equation can be absorbed into the wave function by a local phase redefinition involving the line integral of the vector potential, such that the solution to the Schroedinger equation of an electron in the presence

of a magnetic field is given by equation (6), the product of the wavefunction of a free particle times a phase factor. Therefore, the solutions to the equations (22) are

$$\begin{aligned}\Psi_a(\mathbf{x}, t) &= \phi(\mathbf{x}, t) \exp\left(ie \int_{\mathbf{x}_0}^{\mathbf{x}} A_a\right), \\ \Psi_b(\mathbf{x}, t) &= \phi(\mathbf{x}, t) \exp\left(ie \int_{\mathbf{x}_0}^{\mathbf{x}} A_b\right),\end{aligned}\tag{23}$$

such that, in the region R_{ab} of overlap, where the potentials are related by (21), the wavefunctions are related by

$$\Psi_a(\mathbf{x}) = \exp\left(2ieg \int_{\mathbf{x}_0}^{\mathbf{x}} d\phi\right) \Psi_b(\mathbf{x}) = e^{2ieg\phi} \Psi_b(\mathbf{x})\tag{24}$$

if the angle ϕ at the point \mathbf{x}_0 is taken as 0. If the path over which we perform the integral is a closed, then $\phi = 2\pi$ and the requirement that the wavefunction be single-valued implies that

$$4\pi eg = 2\pi n,\tag{25}$$

with n a natural number. This is again Dirac's quantization condition.

We identify the structure appearing here as that of a complex line bundle over $\mathbf{R}^3 - \{0\}$ with fiber $U(1)$. \mathcal{B} is a 2-form well-defined over $\mathbf{R}^3 - \{0\}$, where it actually satisfies $d\mathcal{B} = 0$. As we learned, \mathcal{B} cannot be globally exact. However, we saw that it can be written as $\mathcal{B} = d\mathcal{A}$ locally, over patches R_a and R_b , with \mathcal{A}_a and \mathcal{A}_b , respectively. In this bundle, we see that the electron's wavefunction Ψ is not a function but a local section over each patch, and, in the overlap between patches, it transforms as a section ought to transform, according to equation (24). Additionally, the 1-form \mathcal{A} , in the overlap between patches, transforms as a local connection 1-form according to equation (21). In this context, Dirac's quantization condition allows this bundle to have well-defined sections.

V | INTEGRATING OVER DIFFERENT PATCHES AND DIRAC'S QUANTIZATION CONDITION

We now review (Alvarez, 1985) by Álvarez, in which a different argumentation leads again to Dirac's quantization condition. He explores the consequences of the path initiated by Wu & Yang, who emphasized that the vector potential associated to a magnetic monopole is well-defined only locally, by acknowledging that in principle one can subdivide space into arbitrary sets of patches, over each of which one defines a local potential. As we will see, this might lead to certain ambiguities in the quantum description of charged particles under the monopole's field, which can only be resolved if again Dirac's quantization condition is imposed.

1 | Line integral over different patches

Consider again an electron in the field of a magnetic monopole, but now the particle is constrained to move on the two-sphere. The lagrangian corresponding to the particle contains a kinetic term, an electromagnetic term and a coupling between the monopole's vector potential and the particle's velocity of the form $A_\mu(dx^\mu/dt)$. This lagrangian was obtained in (Wu and Yang, 1976). Recall that the vector potential is a local *connection* 1-form in the base manifold of a bundle with structure group $U(1)$. It is this coupling that we will concentrate on because of its topological properties. The contribution of this

coupling to the action, for a given evolution, would be

$$\int_{\Gamma} A, \tag{26}$$

the line integral of A along some path Γ in configuration space. However, there is the complication that there does not exist a non-singular vector potential over the entire sphere. The solution is to cover the sphere with different patches over which there is a well defined vector potential. In case of having a path belonging to two different patches, one has to integrate the connection defined in one patch up to some point in the overlap between both patches, then switch to the connection defined on other patch and integrate it there. However, this is not trivial.

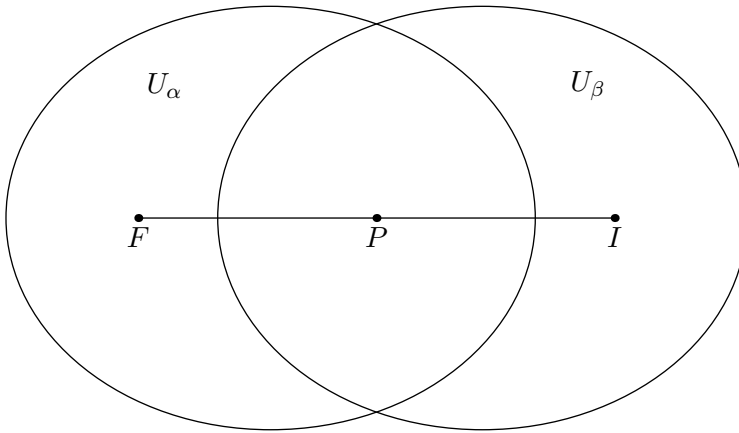


Figure 7: Double overlap. The path comprises two patches and goes from F to I .

Consider Figure 7. There we have such path belonging to two patches. We integrate A_{α} in U_{α} from F to P in the overlap and then A_{β} from P to I in U_{β} . We call

$$I_P = \int_F^P A_{\alpha} + \int_P^I A_{\beta}. \tag{27}$$

This is the integral associated to having chosen the particular point P . We will show that this integral actually depends on P . Consider a different point $Q \in U_{\alpha} \cap U_{\beta}$. We have

$$I_Q = \int_F^Q A_{\alpha} + \int_Q^I A_{\beta}, \tag{28}$$

so that

$$\begin{aligned} I_P - I_Q &= \int_F^P A_{\alpha} + \int_P^I A_{\beta} - \int_F^Q A_{\alpha} - \int_Q^I A_{\beta} \\ &= \int_F^Q A_{\alpha} + \int_Q^P A_{\alpha} + \int_P^I A_{\beta} - \int_F^Q A_{\alpha} - \int_Q^P A_{\beta} - \int_P^I A_{\beta} \\ &= \int_Q^P (A_{\alpha} - A_{\beta}). \end{aligned} \tag{29}$$

The resulting integral is defined over a segment PQ in the intersection $U_{\alpha} \cap U_{\beta}$. Being both connections

defined in the intersection, they are related by a gauge transformation, $A_\alpha = A_\beta + d\psi_{\alpha\beta}$, with $\psi_{\alpha\beta}$ being some function defined in the overlap. Therefore,

$$I_P - I_Q = \int_Q^P (A_\alpha - A_\beta) = \int_Q^P d\psi_{\alpha\beta} = \psi_{\alpha\beta}(P) - \psi_{\alpha\beta}(Q). \tag{30}$$

In particular, the quantity $I = I_P - \psi_{\alpha\beta}(P)$ is independent of the choice of P . We owe this interesting observation to (Wu and Yang, 1976). The missing step was subtracting the gauge function $\psi_{\alpha\beta}$ evaluated at the intermediate point P .

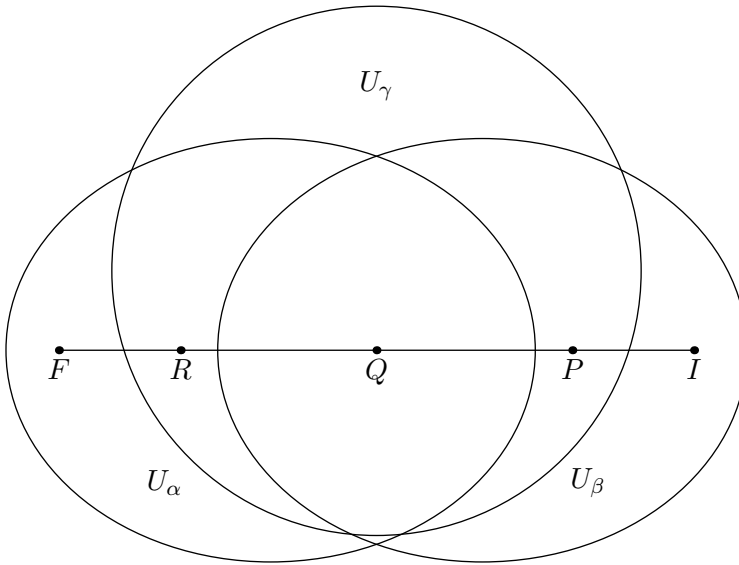


Figura 8: Triple overlap. The path comprises three patches and goes from F to I .

What happens when we have a triple overlap, such as in Figure 8? We could be done with just two patches, but what if we want to consider three patches? In the end, we can partition the whole space into patches in arbitrary ways. Then the line integral is

$$I_3 = \int_F^R A_\alpha - \psi_{\alpha\gamma}(R) + \int_R^P A_\gamma - \psi_{\gamma\beta}(P) + \int_P^I A_\beta. \tag{31}$$

We choose intermediate points R, P in the overlaps between patches, integrate along one patch up to those points, include the contribution by the gauge functions and keep integrating along the other patch and so on. We use the subscript 3 to denote that we have used the three patches. A straightforward calculation, completely analogous to the one for two patches, shows that such prescription is independent of P and Q . This result resembles the one for two patches, we integrate along a certain patch, include the transition function anytime we change patch, keep integrating over the new patch, and so on. We may use instead just two patches, yielding the integral

$$I_2 = \int_F^Q A_\alpha - \psi_{\alpha\beta}(Q) + \int_Q^I A_\beta. \tag{32}$$

with an intermediate point Q . The integrals along FQ and QI , over a single patch, are well defined.

However, these belong partially to U_γ and can be further decomposed. Take points $R \in U_\alpha \cap U_\gamma$, $P \in U_\gamma \cap U_\beta$. Then, since $FQ = FR + RQ$ and $QI = QP + PI$,

$$I_2 = \int_F^R A_\alpha + \int_R^Q A_\alpha - \Psi_{\alpha\beta}(Q) + \int_Q^P A_\beta + \int_P^I A_\beta. \quad (33)$$

There is no cost in splitting because the integrals are performed over the same patch. But $RQ, QP \in U_\gamma$ and we can use the connection defined in U_γ . Since in the corresponding overlaps $A_\alpha = A_\gamma + d\Psi_{\alpha\gamma}$ and $A_\beta = A_\gamma + d\Psi_{\beta\gamma}$, we can substitute that in the last expression, integrate the differentials as to obtain boundary terms, do some algebra and obtain

$$\begin{aligned} I_2 &= \int_F^R A_\alpha - \Psi_{\alpha\gamma}(R) + \int_R^P A_\gamma - \Psi_{\gamma\beta}(R) + \int_P^I A_\beta - [\Psi_{\alpha\beta}(Q) + \Psi_{\beta\gamma}(Q) + \Psi_{\gamma\alpha}(Q)] \\ &= I_3 - [\Psi_{\alpha\beta}(Q) + \Psi_{\beta\gamma}(Q) + \Psi_{\gamma\alpha}(Q)]. \end{aligned} \quad (34)$$

We obtain the result when we use three patches, *plus* some additional terms, all of which are gauge functions evaluated at Q . Observe that

$$\begin{aligned} A_\alpha - A_\beta &= d\Psi_{\alpha\beta}, \\ A_\beta - A_\gamma &= d\Psi_{\beta\gamma}, \\ A_\gamma - A_\alpha &= d\Psi_{\gamma\alpha}, \end{aligned} \quad (35)$$

which, summed, implies that

$$d(\Psi_{\alpha\beta} + \Psi_{\beta\gamma} + \Psi_{\gamma\alpha}) = 0, \quad (36)$$

the extra terms give a closed 2-form. The sum of functions in the last equation, whose differential vanishes, is defined in the triple intersection $U_\alpha \cap U_\beta \cap U_\gamma$. We may use a covering such that this triple intersection is diffeomorphic to an open ball. Then, by the Poincaré lemma, the closed form can be integrated, resulting in

$$\Psi_{\alpha\beta} + \Psi_{\beta\gamma} + \Psi_{\gamma\alpha} = c_{\alpha\beta\gamma}, \quad (37)$$

where $c_{\alpha\beta\gamma}$ is a constant over the triple overlap. In conclusion, we may use two or three patches to evaluate the line integral of the connection and there would be a difference of a constant term. If this line integral appears in the action, this constant term does not modify the classical equations of motion. In quantum mechanics, however, this may lead to ambiguities. The article (Alvarez, 1985) calls this an ambiguity because there is no prescription for a choice of patches, these possible choices differing by a constant. Making this ambiguity physically irrelevant will lead, once again, to Dirac's quantization condition.

2 | Resolving the ambiguity

The wavefunction ψ of an electron under the magnetic field of a monopole is given by $e^{ie \int_\Gamma A} \psi_{\text{free}}$, as we saw, where there is additional phase factor involving a line integral of the vector potential. This term may produce observable consequences in interference experiments, like in the Aharonov-Bohm experiment. There is an ambiguity in the line integral of the connection, for the reasons mentioned above, there is an extra $e^{iec_{\alpha\beta\gamma}}$ phase factor when triple overlaps are considered. But we can make this ambiguity unobservable if each $c_{\alpha\beta\gamma}$ can be made equal to an integer multiple of $2\pi/e$. In the following we will see that this leads precisely to Dirac's quantization condition.

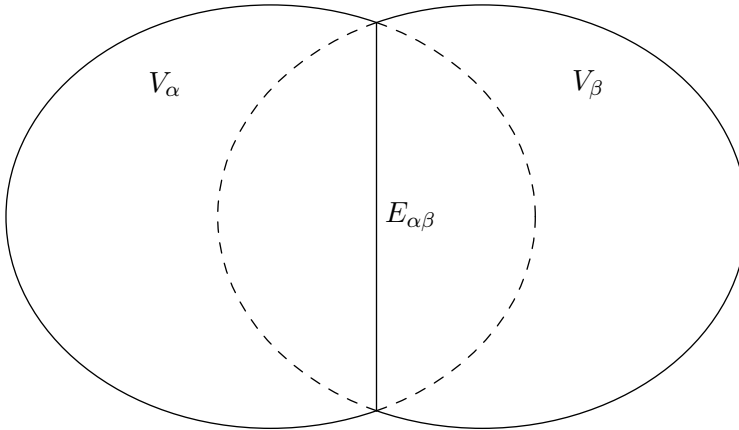


Figura 9: Magnetic flux through a couple of overlapping patches.

In order to relate the constants $c_{\alpha\beta\gamma}$ to the quantization condition, we first integrate the magnetic field B over the entire sphere. If we integrate the magnetic field over all patches, then we are over summing because we are considering several times the overlapping regions. Therefore we consider regions V_α like the ones in Figure 9 which split the overlaps and prevent us from considering them twice. Then the integral of the magnetic field can be taken as

$$\int_{S^2} B = \sum_{\alpha} \int_{V_\alpha} B. \tag{38}$$

In each of the V_α , we have a well behaved vector potential and by using Stokes' theorem we write

$$\sum_{\alpha} \int_{V_\alpha} B = \sum_{\alpha} \int_{V_\alpha} dA = \sum_{\alpha} \int_{\partial V_\alpha} A. \tag{39}$$

The line $E_{\alpha\beta}$ is a common border of V_α and V_β . According to the last expression, the line integral along this border appears twice, once with respect to A_α and once with respect to A_β , with opposite orientations. Therefore there is a contribution to the flux given by the term

$$\int_{E_{\alpha\beta}} (A_\alpha - A_\beta) = \int_{E_{\alpha\beta}} d\Psi_{\alpha\beta} = \int_{\partial E_{\alpha\beta}} \Psi_{\alpha\beta}, \tag{40}$$

since this border belongs to the overlap, where $A_\alpha = A_\beta + d\Psi_{\alpha\beta}$. The last expression is just the value of $\Psi_{\alpha\beta}$ at the endpoints of $E_{\alpha\beta}$.

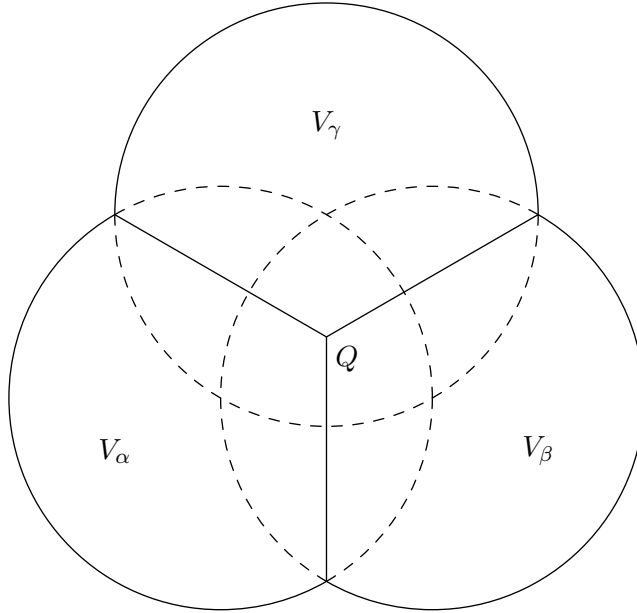


Figure 10: Magnetic flux in a triple of overlapping patches.

What happens when we have triple intersections, like in Figure 10? By the same arguments, the integral of the magnetic field over the three patches will be the integral over the boundaries of \$V_\alpha\$, \$V_\beta\$ and \$V_\gamma\$. Each of these boundaries contributes with two common borders, \$E_{\alpha\beta}\$, \$E_{\beta\gamma}\$ and \$E_{\gamma\alpha}\$, so that these appear twice with opposite orientations. Therefore, there will be a contribution of the form

$$\int_{\partial E_{\alpha\beta}} \Psi_{\alpha\beta} + \int_{\partial E_{\beta\gamma}} \Psi_{\beta\gamma} + \int_{\partial E_{\gamma\alpha}} \Psi_{\gamma\alpha} \tag{41}$$

which will yield the term \$\Psi_{\alpha\beta}(Q) + \Psi_{\beta\gamma}(Q) + \Psi_{\gamma\alpha}(Q)\$, already familiar to us as \$c_{\alpha\beta\gamma}\$ plus the transition functions evaluated at the remaining endpoints of the common boundaries, which, after considering the complementary patches, will be considered themselves points in triple intersections. Therefore, the total magnetic flux will be

$$\int_{S^2} B = \sum_{U_{\alpha\beta\gamma}} c_{\alpha\beta\gamma} = \frac{2\pi}{e} \sum_{U_{\alpha\beta\gamma}} n_{\alpha\beta\gamma} \tag{42}$$

where \$n_{\alpha\beta\gamma}\$ is an integer value such that the constant \$c_{\alpha\beta\gamma} = 2\pi n_{\alpha\beta\gamma}/e\$ over the triple intersection \$U_{\alpha\beta\gamma} = U_\alpha \cap U_\beta \cap U_\gamma\$. We know that \$n_{\alpha\beta\gamma}\$ has to be an integer value as to remove the ambiguity in the wavefunction of an electron after patching the sphere with double or triple overlaps. If the total magnetic flux over \$S^2\$ is \$4\pi g\$, then

$$4\pi e g = 2\pi \sum_{U_{\alpha\beta\gamma}} n_{\alpha\beta\gamma}, \tag{43}$$

which is precisely Dirac's quantization condition. This arises again from a consistent unambiguous quantum mechanics of an electron in a monopole field.

3| Čech cohomology

Čech cohomology is the setting in which the previous discussion can be formalized. Consider a manifold M . On this manifold, we will consider that it is always possible to have an open cover $\{U_\alpha\}$ such that each open set and each non-empty finite intersection of open sets is diffeomorphic to an open ball in \mathbb{R}^n . This is called a *good cover*. The existence of such cover is guaranteed for all manifolds and, when the manifold is compact, this cover can be chosen to be finite (Raoul Bott, 1982). The benefit of this cover is that in each intersection, Poincaré lemma holds. One can dispense of a good cover at the expense of extra technical complications.

On each non-empty finite intersection, we define

$$\begin{aligned} U_{\alpha\beta} &= U_\alpha \cap U_\beta, \\ U_{\alpha\beta\gamma} &= U_\alpha \cap U_\beta \cap U_\gamma, \\ U_{\alpha\beta\gamma\delta} &= U_\alpha \cap U_\beta \cap U_\gamma \cap U_\delta, \end{aligned} \tag{44}$$

and so on. An orientation is formally defined as “ $U_{\alpha\beta} = -U_{\beta\alpha}$ ”. In each of these objects, we define a triangulation. In the interior of each open set, we take a point which will be the vertex of a resulting triangulation: for a single open set, we have a point as 0-simplex, for a double intersection we have a line as 1-simplex, for a triple intersection we have a triangle as 2-simplex and so on. This way, the open cover results in a triangulation of the entire manifold. Additionally, the orientation difference between $U_{\alpha\beta}$ and $U_{\beta\alpha}$ makes sense, they give rise to 1-simplices with opposite orientations. This will be the building blocks of a simplicial homology.

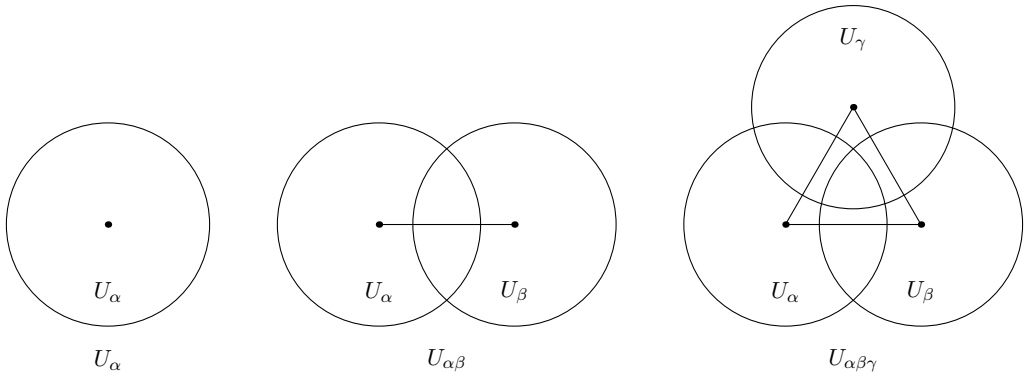


Figure 11: A single open set gives rise to a 0-simplex (a point). Two open sets give rise to a 1-simplex (a line). Three open sets give rise to a 2-simplex (a triangle including the interior).

In general, the object $U_{\alpha_0\alpha_1\dots\alpha_p}$ is called a p -simplex and, from these, we define p -chains as formal linear combinations with integer coefficients of p -simplices. Then, the definition of Čech homology proceeds as a simplicial homology. We call $C_p(\mathcal{U})$ the set of all p -chains. The coboundary operator ∂ is such that $\partial U_{\alpha\beta} = U_\beta - U_\alpha$, $\partial U_{\alpha\beta\gamma} = U_{\beta\gamma} - U_{\alpha\gamma} + U_{\alpha\beta}$ and so on for higher chains. In general, the operator ∂ is such that $\partial : C_p(\mathcal{U}) \rightarrow C_{p-1}(\mathcal{U})$ and acts on a p -simplex as

$$\partial U_{\alpha_0\dots\alpha_p} = \sum_{i=0}^p (-1)^i U_{\alpha_0\dots\hat{\alpha}_i\dots\alpha_p}, \tag{45}$$

where $\hat{}$ over an element means removal. We can compute that the application of the boundary operator on the lowest order cochains gives:

$$\begin{aligned} \partial U_\alpha &= 0, \\ \partial U_{\alpha\beta} &= U_\beta - U_\alpha, \\ \partial U_{\alpha\beta\gamma} &= U_{\alpha\beta} + U_{\beta\gamma} + U_{\gamma\alpha}, \\ \partial U_{\alpha\beta\gamma\delta} &= U_{\beta\gamma\delta} - U_{\alpha\gamma\delta} + U_{\alpha\beta\delta} - U_{\alpha\beta\gamma}. \end{aligned} \tag{46}$$

We have, for example, that $\partial(\partial U_{\alpha\beta\gamma}) = \partial(U_{\alpha\beta} + U_{\beta\gamma} + U_{\gamma\alpha}) = U_\beta - U_\alpha + U_\gamma - U_\beta + U_\alpha - U_\gamma = 0$. In general, using (45), it can be shown that the boundary operator is nilpotent, $\partial^2 = 0$.

Consider a p -chain $U_{\alpha_0 \dots \alpha_p}$. If $\partial U_{\alpha_0 \dots \alpha_p} = 0$, we say that the p -chain is a p -cycle. On the other hand, if there exists a $U_{\alpha_0 \dots \alpha_{p-1}}$ such that $U_{\alpha_0 \dots \alpha_p} = \partial U_{\alpha_0 \dots \alpha_{p-1}}$, then the p -chain is a p -boundary. We call $Z_p(\mathcal{U})$ the set of all p -cycles and $B_p(\mathcal{U})$ the set of all p -boundaries. Alternative definitions are $Z_p(\mathcal{U}) = \text{Ker} \partial C_p(\mathcal{U})$ and $B_p(\mathcal{U}) = \text{Im} \partial C_{p-1}(\mathcal{U})$. Since the boundary operator is nilpotent, any p -boundary is a p -cycle, so that $B_p(\mathcal{U}) \subset Z_p(\mathcal{U})$, but not the other way around, not all p -cycles are p -boundaries. Finally, we define the p -th Čech homology group as the quotient $H_p(\mathcal{U}) = Z_p(\mathcal{U})/B_p(\mathcal{U})$. What we have defined so far is the Čech homology, but what will be useful for properly stating the ambiguity in using different patches for the monopole problem is rather Čech cohomology. Here, we define a p -cochain with values in q -forms as an assignment of a local q -form for each p -chain $U_{\alpha_0 \dots \alpha_p}$ associated to the cover \mathcal{U} . We write as $C^p(\mathcal{U}, \Omega^q)$ the set of p -cochains with values in q -forms, where Ω^q is the space of q -forms. For example, the collection $\{A_\alpha\}$ is an element of $C^0(\mathcal{U}, \Omega^1)$, an assignment of a 1-form to each open set; similarly, the collection $\{c_{\alpha\beta}\}$ is an element of $C^2(\mathcal{U}, \Omega^0)$, that is, an assignment of a function on each triple overlap, and so on. Note that p -cochains have $p + 1$ indices. We further define a coboundary operator δ such that $\delta : C^p(\mathcal{U}, \Omega^q) \rightarrow C^{p+1}(\mathcal{U}, \Omega^q)$ and acts in the following way for low p ,

$$\begin{aligned} \delta\{A_\alpha\} &= \{A_\alpha - A_\beta\}, \\ \delta\{B_{\alpha\beta}\} &= \{B_{\alpha\beta} + B_{\beta\gamma} + B_{\gamma\delta}\}, \\ \delta\{C_{\alpha\beta\gamma}\} &= \{C_{\alpha\beta\gamma} + C_{\beta\gamma\delta} + C_{\gamma\delta\alpha} + C_{\delta\alpha\beta}\}. \end{aligned} \tag{47}$$

Given a 0-cochain $\{A_\alpha\}$, we have that $\delta\{A_\alpha\} = \{A_\alpha - A_\beta\}$. That this 0-cochain be a 0-cocycle means that $\{A_\alpha - A_\beta\} = \{0\}$ in each overlap $U_\alpha \cap U_\beta$. This is the condition that A be a global form. Global forms can be identified with $C^{-1}(\mathcal{U}, \Omega^q)$ and for these δ acts just by restricting the global form to each open set. It can be shown that the coboundary operator is nilpotent and therefore we have all we need in order to define cocycles, coboundaries and cohomology groups in exactly the same way as we have done before. However, these are not the cohomology groups that we are not going to refer to these as the Čech cohomology groups.

Thus, we learn that in the monopole problem $\{c_{\alpha\beta\gamma}\} = \delta\{\psi_{\alpha\beta}\}$, that is, the collection of $\{c_{\alpha\beta\gamma}\}$ is a 2-cocycle.

Note that cochains $C^p(\mathcal{U}, \Omega^q)$ are indexed by two labels, the order p or the cochain and the order q of the form. On the space of forms we also have a coboundary operator d , the exterior derivative, raising the order of differential forms and being nilpotent. Therefore, we can also define a cohomology with respect to these operator by restricting to cochains of the same order and forms of the same order. This is the usual de Rham cohomology.

We therefore introduce the tic-tac-toe boxes, with p associated to rows and q to columns. Schematically,

we have

$$\begin{array}{c|cccc}
 \Omega^3 & C^0(u, \Omega^3) & C^1(u, \Omega^3) & C^2(u, \Omega^3) & C^3(u, \Omega^3) \\
 \Omega^2 & C^0(u, \Omega^2) & C^1(u, \Omega^2) & C^2(u, \Omega^2) & C^3(u, \Omega^2) \\
 \Omega^1 & C^0(u, \Omega^1) & C^1(u, \Omega^1) & C^2(u, \Omega^1) & C^3(u, \Omega^1) \\
 \Omega^0 & C^0(u, \Omega^0) & C^1(u, \Omega^0) & C^2(u, \Omega^0) & C^3(u, \Omega^0) \\
 \hline
 d \uparrow & & & & \\
 \delta \rightarrow & C^0 & C^1 & C^2 & C^3
 \end{array} \tag{48}$$

In the monopole problem, the initial data are the vector potentials $\{A_\alpha\}$ locally defined in each patch and the transition functions $\{\Psi_{\alpha\beta}\}$ in each overlap between patches. We move vertically and horizontally with the coboundary operators associated to both cohomologies in the following way:

$$\begin{array}{c|cccc}
 \Omega^3 & 0 & & & \\
 \Omega^2 & \{dA_\alpha\} = \{F_\alpha\} & 0 & & \\
 \Omega^1 & \{A_\alpha\} & \delta\{A_\alpha\} = \{d\Psi_{\alpha\beta}\} & 0 & \\
 \Omega^0 & & \{\Psi_{\alpha\beta}\} & \delta\{\Psi_{\alpha\beta}\} = \{c_{\alpha\beta\gamma}\} & 0 \\
 \hline
 d \uparrow & & & & \\
 \delta \rightarrow & C^0 & C^1 & C^2 & C^3
 \end{array} \tag{49}$$

From this, we learn that F is d -closed and zero δ -cocycle. Therefore F is a global closed differential form. Additionally, we learn that $\{c_{\alpha\beta\gamma}\}$ is a d -closed 0-form two δ -cocycle.

VI | THREE-COCYCLE IN QUANTUM MECHANICS

There exists yet another approach to the quantization condition found by Dirac, which we owe to Jackiw (Jackiw, 1985). This other argumentation is still more surprising, because the violation of the quantization condition implies that the quantum mechanics of an electron under a monopole field becomes non-associative. Different ways of composing three successive translation operators will produce results differing in a phase factor that will be interpreted as a certain cochain in a particular cohomology.

1 | Non-associativity and Jacobi identity violation in Quantum Mechanics

Consider an electron in the background magnetic field \mathbf{B} of a monopole located at position \mathbf{r}_0 . For simplicity, we set $m = \hbar = 1$. The magnetic field satisfies

$$\nabla \cdot \mathbf{B} = 4\pi g \delta(\mathbf{x}), \tag{50}$$

where g is the monopole's charge. In this case, the canonical momentum conjugate to \mathbf{x} , is

$$\mathbf{p} = \mathbf{v} + e\mathbf{A} \tag{51}$$

where \mathbf{v} is the electron's velocity and \mathbf{A} is the vector potential that satisfies $\nabla \times \mathbf{A} = \mathbf{B}$. As we know, the momentum \mathbf{p} is gauge dependent, while \mathbf{v} is gauge independent, it is just the time derivative of position. Since \mathbf{A} is only a function of position \mathbf{x} , both \mathbf{p} and \mathbf{v} have the same commutation relations with \mathbf{x} , and we can generate spatial translations with either of them. If we are interested in gauge invariance, we use the gauge invariant operator \mathbf{v} , so that the gauge invariant representation of a translation by a vector \mathbf{a} is given by $U(\mathbf{a}) = e^{i\mathbf{a}\cdot\mathbf{v}}$, since we can easily show, using Baker-Campbell-Hausdorff, that $U(\mathbf{a})\mathbf{x}U^{-1}(\mathbf{a}) = \mathbf{x} + \mathbf{a}$. However, this representation is not trivial since \mathbf{v} satisfies the algebra

$$[v^i, v^j] = ie \sum_k \epsilon^{ijk} B^k. \tag{52}$$

This is the usual story about the non-commutativity induced by the presence of a magnetic field. Furthermore, we can see that

$$[[v^1, v^2], v^3] + [[v^3, v^1], v^2] + [[v^2, v^3], v^1] = -4\pi eg \delta(\mathbf{r} - \mathbf{r}_0). \tag{53}$$

This means that the invariant generators of translations satisfy the Jacobi identity everywhere except at the monopole's position. This happens at the level of translation generators. Let's see what happens at the group representation level.

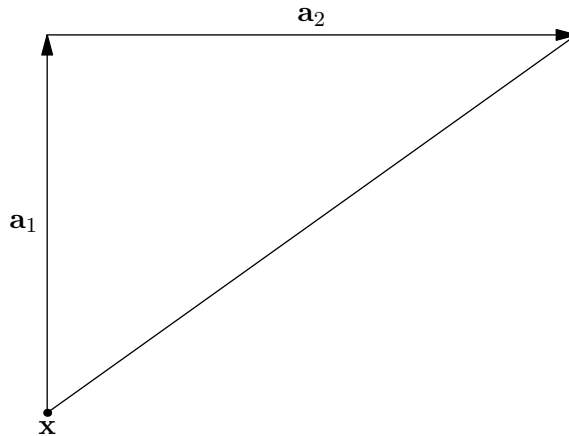


Figura 12: The phase in the projective product is the magnetic flux across this triangle.

Given a wavefunction $\Psi(\mathbf{x})$ in the coordinate representation, the action of the translation operator can

be expressed as

$$U(\mathbf{a})\Psi(\mathbf{x}) = e^{i\mathbf{a}\cdot\mathbf{v}}\Psi(\mathbf{x}) = e^{i\mathbf{a}\cdot\mathbf{v}}e^{-i\mathbf{a}\cdot\mathbf{p}}\Psi(\mathbf{x} + \mathbf{a}) = \exp\left(-ie\int_{\mathbf{x}}^{\mathbf{x}+\mathbf{a}}\mathbf{A}\cdot d\mathbf{s}\right)\Psi(\mathbf{x} + \mathbf{a}) \quad (54)$$

with the line integral being performed along the *straight* line joining the points \mathbf{x} and $\mathbf{x} + \mathbf{a}$. The easiest way to prove this result is by defining the operator $W(\lambda) = e^{i\lambda\mathbf{a}\cdot\mathbf{v}}e^{-i\lambda\mathbf{a}\cdot\mathbf{p}}$, which can be shown to satisfy the differential equation $dW(\lambda)/d\lambda = -ie\mathbf{a}\cdot\mathbf{A}(\mathbf{x} + \lambda\mathbf{a})W(\lambda)$, with $W(0) = 1$. A solution of this differential equation is $W_1(\lambda) = \exp\left(-ie\int_0^\lambda\mathbf{A}(\mathbf{x} + s\mathbf{a})\cdot\mathbf{a}ds\right) = \exp\left(-ie\int_{\mathbf{x}}^{\mathbf{x}+\lambda\mathbf{a}}\mathbf{A}\cdot d\mathbf{s}\right)$, but due to uniqueness of solutions, $W(\lambda) = W_1(\lambda)$ and the result follows by evaluating at $\lambda = 1$. Thus the action of the translation operator carries an additional phase. For a product of two translation operators, we have

$$\begin{aligned} U(\mathbf{a}_1)U(\mathbf{a}_2)\Psi(\mathbf{x}) &= e^{i\mathbf{a}_1\cdot\mathbf{v}}e^{i\mathbf{a}_2\cdot\mathbf{v}}\Psi(\mathbf{x}) \\ &= \exp\left(-ie\int_{\mathbf{x}}^{\mathbf{x}+\mathbf{a}_1}\mathbf{A}\cdot d\mathbf{s}\right)\exp\left(-ie\int_{\mathbf{x}+\mathbf{a}_1}^{\mathbf{x}+\mathbf{a}_1+\mathbf{a}_2}\mathbf{A}\cdot d\mathbf{s}\right)\Psi(\mathbf{x} + \mathbf{a}_1 + \mathbf{a}_2) \\ &= \exp\left(-ie\int_{\mathbf{x}}^{\mathbf{x}+\mathbf{a}_1}\mathbf{A}\cdot d\mathbf{s}\right)\exp\left(-ie\int_{\mathbf{x}+\mathbf{a}_1}^{\mathbf{x}+\mathbf{a}_1+\mathbf{a}_2}\mathbf{A}\cdot d\mathbf{s}\right)\exp\left(-ie\int_{\mathbf{x}+\mathbf{a}_1+\mathbf{a}_2}^{\mathbf{x}}\mathbf{A}\cdot d\mathbf{s}\right) \\ &\quad \times U(\mathbf{a}_1 + \mathbf{a}_2)\Psi(\mathbf{x}) \\ &= \exp\left(-ie\int_{\Delta}\mathbf{B}\cdot d\mathbf{n}\right)U(\mathbf{a}_1 + \mathbf{a}_2)\Psi(\mathbf{x}) \end{aligned} \quad (55)$$

where Δ is the triangle formed by the vertices $(\mathbf{x}, \mathbf{x} + \mathbf{a}_1, \mathbf{x} + \mathbf{a}_1 + \mathbf{a}_2)$ with surface element $d\mathbf{n}$ along the outward direction $\mathbf{a}_1 \times \mathbf{a}_2$. This integral is therefore the magnetic flux across the surface Δ . The second equality makes evident that the representation of translations is projective. The last equality shows that the composition of two translation operators produces a phase factor related to the magnetic flux through a triangle Δ generated by the translation directions.

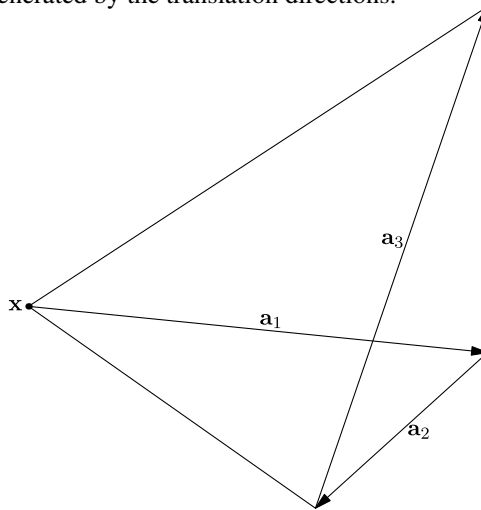


Figura 13: The phase appearing in the non-associative product is the magnetic flux across this tetrahedron.

It is interesting that from the product of two translations we can derive the product of three translations, namely,

$$U(\mathbf{a}_1)[U(\mathbf{a}_2)U(\mathbf{a}_3)] = e^{-ie\Phi_{23}}U(\mathbf{a}_1)U(\mathbf{a}_2 + \mathbf{a}_3) = e^{-ie(\Phi_{23} + \Phi_{1(2+3)})}U(\mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3) \quad (56)$$

$$[U(\mathbf{a}_1)U(\mathbf{a}_2)]U(\mathbf{a}_3) = e^{-ie\Phi_{12}}U(\mathbf{a}_1 + \mathbf{a}_2)U(\mathbf{a}_3) = e^{-ie(\Phi_{12} + \Phi_{(1+2)3})}U(\mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3), \quad (57)$$

so that

$$[U(\mathbf{a}_1)U(\mathbf{a}_2)]U(\mathbf{a}_3) = e^{-ie(\Phi_{23} + \Phi_{1(2+3)} - \Phi_{12} - \Phi_{(1+2)3})}U(\mathbf{a}_1)[U(\mathbf{a}_2)U(\mathbf{a}_3)], \quad (58)$$

where Φ_{ij} is the magnetic flux across the triangle with vertices $(\mathbf{x}, \mathbf{x} + \mathbf{a}_i, \mathbf{x} + \mathbf{a}_i + \mathbf{a}_j)$ with outward normal in the direction of $\mathbf{a}_i \times \mathbf{a}_j$ and $\Phi_{(i+j)k}$ is the magnetic flux across the triangle with vertices $(\mathbf{x}, \mathbf{x} + \mathbf{a}_i + \mathbf{a}_j, \mathbf{x} + \mathbf{a}_i + \mathbf{a}_j + \mathbf{a}_k)$ with outward normal in the direction of $(\mathbf{a}_i + \mathbf{a}_j) \times \mathbf{a}_k$ and so on. Considering the tetrahedron in Figure 13, we may identify the normal outward orientation of its faces, so that Φ_{23} goes inward, Φ_{12} goes outward, $\Phi_{1(2+3)}$ goes inward and $\Phi_{(1+2)3}$ goes outward. Therefore, $\Phi_{12} - \Phi_{23} + \Phi_{(1+2)3} - \Phi_{1(2+3)}$ is the total outward flux across the tetrahedron's faces and the exponent in the non-associative product is e times this total flux. If the tetrahedron does not enclose the monopole, then this flux vanishes and associativity is restored. However, if the tetrahedron encloses the monopole, there is a net flux equal to $4\pi g$ and a total phase of $4\pi eg$. As we shrink the tetrahedron to a point, we obtain infinitesimally the violation of the Jacobi identity. Note that if we wish to restore associativity, then eg must be quantized in units of $1/2$, which is Dirac's quantization condition. In the following we will see that the phase appearing in the non-associative product corresponds to a 3-cocycle in an appropriate cohomology.

2 | Cohomology for group actions on manifolds

The following discussion is based faithfully on section 5.4 of (Azcárraga and Izquierdo, 1995). Defining a cohomology involves defining *cochains* and a nilpotent *coboundary* operator. The proper context is that of a group G acting on a manifold \mathcal{M} , with the cochains being functions on this manifold, that is, elements of $\mathcal{F}(\mathcal{M})$. Addition of functions on the manifold is defined in the natural way, $(f_1 + f_2)(x) = f_1(x) + f_2(x)$. The action of a group element g on a point x in the manifold is written as x^g .

We first define $\mathcal{F}(\mathcal{M})$ -valued n -cochains on G in the following way. Consider a group G acting on a manifold \mathcal{M} . $\mathcal{F}(\mathcal{M})$ -valued n -cochains are mappings $\Omega_n : G \times \dots \times G \rightarrow \mathcal{F}(\mathcal{M})$ so that $\Omega_n : (g_1, \dots, g_n) \mapsto \Omega_n(\cdot; g_1, \dots, g_n)$ where $\Omega_n(\cdot; g_1, \dots, g_n) : \mathcal{M} \rightarrow \mathbb{R}$ and $\Omega_n(\cdot; g_1, \dots, g_n) : x \mapsto \Omega_n(x; g_1, \dots, g_n)$. That is, a n -cochain takes n elements in the group G and returns a function on the manifold \mathcal{M} that uses those elements as parameters. The space of n -cochains is $C^n(G, \mathcal{F}(\mathcal{M}))$. The addition of n -cochains Ω_n and Ω'_n is defined by $(\Omega_n + \Omega'_n)(\cdot; g_1, \dots, g_n) = \Omega_n(\cdot; g_1, \dots, g_n) + \Omega'_n(\cdot; g_1, \dots, g_n)$. Thus $C^n(G, \mathcal{F}(\mathcal{M}))$ is an abelian group. Furthermore, $C^0(G, \mathcal{F}(\mathcal{M})) = \mathcal{F}(\mathcal{M})$. We further define the coboundary operator $\delta : C^n(G, \mathcal{F}(\mathcal{M})) \rightarrow C^{n+1}(G, \mathcal{F}(\mathcal{M}))$ such that

$$\begin{aligned} [(\delta\Omega_n)(\cdot; g_1, \dots, g_{n+1})](x) &= (\delta\Omega_n)(x; g_1, \dots, g_{n+1}) \\ &\equiv (-1)^{n+1}\Omega_n(x; g_1, \dots, g_n) \\ &\quad + \sum_{i=1}^n (-1)^i \Omega_n(x; g_1, \dots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \dots, g_{n+1}) \\ &\quad + \Omega_n(x^{g_1}; g_2, \dots, g_{n+1}). \end{aligned} \quad (59)$$

The coboundary operator is a mapping between n -cochains to $(n + 1)$ -cochains, that is, a mapping between functions on the manifold. We use the extra group element in $\delta\Omega_n$ to define an action Φ of G on $\mathcal{F}(\mathcal{M})$, $\Phi(g) : \mathcal{F}(\mathcal{M}) \rightarrow \mathcal{F}(\mathcal{M})$ such that, given a function f in $\mathcal{F}(\mathcal{M})$, it is satisfied $(\Phi(g)f)(gx) = f(x)$. It follows that $\Phi(g')(\Phi(g)f)(x) = (\Phi(g)f)(g'^{-1}x) = f(g^{-1}g'^{-1}x) = f((g'g)^{-1}x) = (\Phi(g'g)f)(x)$, so that Φ is a left action. It is possible to define a coboundary operator as a *right* action, but we do not consider that possibility since the group we are interested in is abelian. The first applications of the coboundary operator are

$$\begin{aligned} (\delta\Omega_0)(x; g_1) &= -\Omega_0(x) + \Omega_0(x^{g_1}), \\ (\delta\Omega_1)(x; g_1, g_2) &= \Omega_1(x; g_1) - \Omega_1(x; g_1g_2) + \Omega_1(x^{g_1}; g_2), \\ (\delta\Omega_2)(x; g_1, g_2, g_3) &= -\Omega_2(x; g_1, g_2) - \Omega_2(x; g_1g_2, g_3) + \Omega_2(x; g_1, g_2g_3) \\ &\quad + \Omega_2(x^{g_1}; g_2, g_3), \\ (\delta\Omega_3)(x; g_1, g_2, g_3, g_4) &= \Omega_3(x; g_1, g_2, g_3) - \Omega_3(x; g_1g_2, g_3, g_4) + \Omega_3(x; g_1, g_2g_3, g_4) \\ &\quad - \Omega_3(x; g_1, g_2, g_3g_4) + \Omega_3(x^{g_1}; g_2, g_3, g_4). \end{aligned}$$

For the previous examples, one can see that $\delta(\delta\Omega_n) = 0$. The general statement that the coboundary operator is nilpotent, $\delta^2 = 0$, can also be proved through straightforward calculations, which is the remaining requirement to be satisfied by the coboundary operator to form a cohomology.

Given a cochain Ω_n , if $\delta\Omega_n = 0$, then the cochain is said to be a cocycle. If there exists some Ω'_{n-1} such that $\delta\Omega'_{n-1} = \Omega_n$, then the cochain is said to be a coboundary. A coboundary is a cocycle because $\delta\Omega_n = \delta(\delta\Omega'_{n-1}) = 0$, but not all cocycles are coboundaries, that is, not because $\delta\Omega_n = 0$ can we conclude that Ω_n is a coboundary. The set of n -cocycles is $Z^n(G, \mathcal{F}(\mathcal{M}))$, while the set of n -coboundaries is $B^n(G, \mathcal{F}(\mathcal{M}))$. Then $B^n(G, \mathcal{F}(\mathcal{M})) \subset Z^n(G, \mathcal{F}(\mathcal{M}))$. When a cocycle turns out to be a coboundary, we say that the cocycle is *trivial*. Otherwise, the structure of non-trivial cocycles is encoded in the cohomology groups. The n -th cohomology group is defined as $H^n(G, \mathcal{F}(\mathcal{M})) = Z^n(G, \mathcal{F}(\mathcal{M}))/B^n(G, \mathcal{F}(\mathcal{M}))$. The underlying equivalence relation is that of two n -cocycles differing by a n -coboundary, that is, given $Z_n, Z'_n \in Z^n(G, \mathcal{F}(\mathcal{M}))$, $Z_n \sim Z'_n$ iff $Z_n - Z'_n = \delta\Omega_{n-1}$ for some $\Omega_{n-1} \in C^{n-1}(G, \mathcal{F}(\mathcal{M}))$.

3 | Cohomology in the monopole problem

In the context discussed previously of a charged particle in magnetic monopole field, we have the abelian group of translations acting on 3-dimensional euclidean spacetime, that is, $G = \mathbb{R}^3$ and $\mathcal{M} = \mathbb{R}^3$. Given any reference point \mathbf{x}_0 , we define a 0-cochain as

$$\Omega_0(\mathbf{x}) = \int_{\mathbf{x}_0}^{\mathbf{x}} \mathbf{A} \cdot d\mathbf{s}. \tag{60}$$

3.1 | 1-cochain

Defined as

$$\Omega_1(\mathbf{x}; \mathbf{a}_1) = \int_{\mathbf{x}}^{\mathbf{x}+\mathbf{a}_1} \mathbf{A} \cdot d\mathbf{s}, \tag{61}$$

the phase (54) appearing in the action of the invariant translations is a 1-cochain. We observe that

$$\begin{aligned} \Omega_1(\mathbf{x}; \mathbf{a}_1) &= \int_{\mathbf{x}}^{\mathbf{x}_0} \mathbf{A} \cdot d\mathbf{s} + \int_{\mathbf{x}_0}^{\mathbf{x}+\mathbf{a}_1} \mathbf{A} \cdot d\mathbf{s} \\ &= -\Omega_0(\mathbf{x}) + \Omega_0(\mathbf{x} + \mathbf{a}_1) \\ &= (\delta\Omega_0)(\mathbf{x}; \mathbf{a}_1). \end{aligned} \tag{62}$$

Therefore, Ω_1 is a trivial cochain.

3.2 | 2-cochain

Defined as the surface integral

$$\Omega_2(\mathbf{x}; \mathbf{a}_1, \mathbf{a}_2) = \int_{\Delta} \mathbf{B} \cdot d\mathbf{n} \tag{63}$$

over the triangle with vertices $(\mathbf{x}, \mathbf{x} + \mathbf{a}_1, \mathbf{x} + \mathbf{a}_1 + \mathbf{a}_2)$ with normal \mathbf{n} in the direction of $\mathbf{a}_1 \times \mathbf{a}_2$, the phase (55) appearing in the projective representation of the invariant translation operator is a 2-cochain. However, note that the boundary of this triangle is given by the straight segments $(\mathbf{x}, \mathbf{x} + \mathbf{a}_1) + (\mathbf{x} + \mathbf{a}_1, \mathbf{x} + \mathbf{a}_1 + \mathbf{a}_2) + (\mathbf{x} + \mathbf{a}_1 + \mathbf{a}_2, \mathbf{x})$, so that, by using Stokes theorem, we have

$$\begin{aligned} \Omega_2(\mathbf{x}; \mathbf{a}_1, \mathbf{a}_2) &= \int_{\partial\Delta} \mathbf{A} \cdot d\mathbf{s} \\ &= \int_{\mathbf{x}}^{\mathbf{x}+\mathbf{a}_1} \mathbf{A} \cdot d\mathbf{s} + \int_{\mathbf{x}+\mathbf{a}_1}^{\mathbf{x}+\mathbf{a}_1+\mathbf{a}_2} \mathbf{A} \cdot d\mathbf{s} + \int_{\mathbf{x}+\mathbf{a}_1+\mathbf{a}_2}^{\mathbf{x}} \mathbf{A} \cdot d\mathbf{s} \\ &= \Omega_1(\mathbf{x}; \mathbf{a}_1) - \Omega_1(\mathbf{x}; \mathbf{a}_1 + \mathbf{a}_2) + \Omega_1(\mathbf{x} + \mathbf{a}_1; \mathbf{a}_2) \\ &= (\delta\Omega_1)(\mathbf{x}; \mathbf{a}_1, \mathbf{a}_2). \end{aligned} \tag{64}$$

However, the use of Stokes' theorem is limited by the presence of the string singularity. Only when the string singularity is not enclosed by Δ can we consider this 2-cochain as trivial. But also in that case, $\Omega_2 = \delta^2\Omega_0$ and the 2-cochain would vanish.

3.3 | 3-cochain

Finally, defined as

$$\Omega_3(\mathbf{x}; \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) = \int_T \mathbf{B} \cdot d\mathbf{n} \tag{65}$$

where T is the tetrahedron defined by the vertices $(\mathbf{x}, \mathbf{x} + \mathbf{a}_1, \mathbf{x} + \mathbf{a}_1 + \mathbf{a}_2, \mathbf{x} + \mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3)$, the phase (58) breaking non-associativity is a 3-cochain. This 3-cochain is the total magnetic flux through the tetrahedron T . If Δ_{ij} is the triangle with vertices $(\mathbf{x}, \mathbf{x} + \mathbf{a}_i, \mathbf{x} + \mathbf{a}_i + \mathbf{a}_j)$ and $\Delta_{i(j+k)}$ is the triangle with vertices $(\mathbf{x}, \mathbf{x} + \mathbf{a}_i, \mathbf{x} + \mathbf{a}_i + \mathbf{a}_j + \mathbf{a}_k)$, we can write this total flux as a sum of outward fluxes through each

of the four tetrahedron's faces as

$$\begin{aligned}
 \Omega_3(\mathbf{x}; \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) &= \int_{\Delta_{12}} \mathbf{B} \cdot d\mathbf{n} + \int_{\Delta_{32}} \mathbf{B} \cdot d\mathbf{n} + \int_{\Delta_{(12)3}} \mathbf{B} \cdot d\mathbf{n} + \int_{\Delta_{(23)1}} \mathbf{B} \cdot d\mathbf{n} \\
 &= \int_{\Delta_{12}} \mathbf{B} \cdot d\mathbf{n} + \int_{\Delta_{(12)3}} \mathbf{B} \cdot d\mathbf{n} - \int_{\Delta_{23}} \mathbf{B} \cdot d\mathbf{n} - \int_{\Delta_{1(23)}} \mathbf{B} \cdot d\mathbf{n} \\
 &= \Omega_2(\mathbf{x}; \mathbf{a}_1, \mathbf{a}_2) + \Omega_2(\mathbf{x}; \mathbf{a}_1 + \mathbf{a}_2, \mathbf{a}_3) - \Omega_2(\mathbf{x}; \mathbf{a}_1, \mathbf{a}_2 + \mathbf{a}_3) - \Omega_2(\mathbf{x} + \mathbf{a}_1; \mathbf{a}_2, \mathbf{a}_3) \\
 &= -(\delta\Omega_2)(\mathbf{x}; \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3).
 \end{aligned} \tag{66}$$

We have repeatedly used that Δ_{ij} and Δ_{ji} have opposite orientations and therefore integrals over such triangles have opposite signs. This last equality shows that Ω_3 is also trivial. In the case that Ω_2 was trivial, then $\Omega_3 = \delta^2\Omega_1 = 0$. That was determined by the faces of the tetrahedron being penetrated by Dirac's string.

As Jackiw points out in (Jackiw, 1984), that all these cocycles be trivial implies that an operator redefinition can be done that restores a trivial representation and associativity. This representation is, no surprise, the one generated by the gauge dependent \mathbf{p} . Nevertheless, if we stick to representing translations with gauge invariant operators, we must remain with these cocycles.

VII | DISCUSSION

In this short review, we have presented some of the arguments leading to the Dirac quantization condition. There exist yet other arguments, which we invite the reader to consult on his own (Deguchi and Kitsukawa, 2006). What we learn in the end is that if magnetic monopoles are ever found to exist, the quantization condition has to be satisfied in order to avoid a catastrophe in the so far extremely effective mathematical structure of fundamental physics: gauge invariance and associativity. At the core of the arguments stands out the impossibility to extend differential forms over the whole of space due to the non-trivial topology of configuration space that a monopole induces. This is what the de Rham and the Čech cohomologies are telling us.

It remains to explore the equivalence or the interconnections between the different arguments, which we have not attempted to do and which might prove to be a very fruitful endeavor.

Although these descriptions are elegant and perhaps paradigmatic in the clean and efficient application of mathematics to physics, in this case as an explanation of the quantization of electric charge, Nature is yet to reveal if these notions are of any relevance here. It is pertinent here to bring the sober attitude of Dirac himself (Dirac, 1981) (quoted from (Jackiw, 2004)):

I am inclined now to believe that monopoles do not exist. So many years have gone by without any encouragement from the experimental side.

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