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La formulación de las ecuaciones de Maxwell a través del bi-complejo variacional

The variational bi-complex formulation of Maxwell's equations

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In this manuscript, we present the variational derivation of Maxwell's equations by means of the variational bi-complex. We use this exercise to introduce the reader to the formulation of geometric variational problems and their implementation in a Computer Algebra System.

En este trabajo presentamos la derivación variacional de las ecuaciones de Maxwell por medio del bi-complejo variacional. Usamos este ejercicio para introducir al lector a la formulación geométrica de problemas variacionales y su implementación en un Sistema de Algebra Computacional.

PALABRAS CLAVES

Electromágnetismo, Complejo Variacional, Sistema de Álgebra Computacional

KEYWORDS

Electromagnetism, Variational Complex, Computer Algebra System

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I | THE VECTOR FORMULATION OF MAXWELL FIELD EQUATIONS

Electromagnetism is perhaps the first encounter a student has with a classical field theory. Its empirical evidence lies on the fact that in nature there is a distinguished property of matter that some objects posses and which we call electric charge. Such property is observed to be conserved and it can be quantified in terms of a dynamical law. Accordingly, from the dynamical law we infer the existence of a field responsible for the inertial change of the charges and, in turn, as charges move around a new field configuration arises. Thus, the field itself obeys a dynamical law of its own which is linked to the dynamics of the charges in space.

Historically, the above picture was developed through a set of independent observations about the work done by the fields on test charges and currents, and the fluxes across known surfaces. Recall that the definition of the work done by a vector field $\vec{X}$ along a curve γ is

$$\varepsilon_\gamma(\vec{X}) \equiv \int_\gamma \vec{X} \cdot \mathrm{d}\ell, \tag{1}$$

whilst the flux of a vector field $\vec{X}$ across a given surface Σ is

$$\Phi_\Sigma(\vec{X}) \equiv \int_\Sigma \vec{X}\cdot \mathrm{d}s. \tag{2}$$

The field $\vec{X}$ is said to be *conservative* if the work done (1) is path independent or, equivalently, if the work done along *any* closed path is identically zero. Thus, in its original form, electromagnetism was formulated in terms of global relations between the fields and their sources, namely

$$\oint_{\partial\Omega} \vec{B}\cdot \mathrm{d}s = 0 \quad \text{[Magnetic Gauss' Law]}, \tag{3}$$

$$\oint_{\partial\Sigma} \vec{E}\cdot \mathrm{d}\ell = -\frac{\mathrm{d}}{\mathrm{d}t}\int_\Sigma \vec{B}\cdot \mathrm{d}s \quad \text{[Faraday's Law]}, \tag{4}$$

$$\oint_{\partial\Omega} \vec{D}\cdot \mathrm{d}s = \int_\Omega \rho_{\text{ext}}\mathrm{d}v \quad \text{[Electric Gauss' Law]} \quad \text{and} \tag{5}$$

$$\oint_{\partial\Sigma} \vec{H}\cdot \mathrm{d}\ell = \frac{\mathrm{d}}{\mathrm{d}t}\int_\Sigma \vec{D}\cdot \mathrm{d}s + \int_\Sigma \vec{j}_{\text{ext}}\cdot \mathrm{d}s \quad \text{[Maxwell-Ampere's Law]}. \tag{6}$$

There are a number of symbols that require a proper introduction. Firstly, note that we have arranged things so that in the left hand side (lhs) of each equation there is only a closed integral alternating between surface and contour. Such alternation is not accidental, since the order is disposed in terms of the fields involved. Thus, the first two equations *link* the magnetic flux $\vec{B}$ with the electric field $\vec{E}$, whilst the latter group links the electric flux $\vec{D}$, the magnetic field $\vec{H}$ and the *external* sources of the fields: the electric charge density ρ_{ext} and current flux $\vec{j}_{\text{ext}}$.
Equations (3 - 6) reveal the geometric nature of each constituent of the theory and allows us to correctly identify fields and fluxes. Note that neither of the fields $\vec{E}$ or $\vec{H}$ are conservative! This is the key principle for the energy conversion process driving our modern society. In the case of a closed path, a better name for the integral (1) is *circulation*. Thus, in our discussion the relevant quantities are the fluxes of B and D and the circulations of E and H. This will be the simple motivation for introducing *differential forms* in the next section.
The passing from the global representation to the local expressions of Maxwell's equations is a straightforward application of the vector calculus integral theorems.Thus, (3-6) become

$$\nabla\cdot\vec{B} = 0, \tag{7}$$

$$\nabla\times\vec{E} = -\frac{\partial}{\partial t}\vec{B}, \tag{8}$$

$$\nabla\cdot\vec{D} = \rho_{\text{ext}} \tag{9}$$

and

$$\nabla\times\vec{H} = \frac{\partial}{\partial t}\vec{D} + \vec{j}_{\text{ext}} \tag{10}$$

Note that an immediate consequence of this local form is a continuity equation for the sources. That is, applying the divergence operator and substituting (9) into (10) it follows that

$$\frac{\partial}{\partial t}\rho_{\text{ext}} + \nabla\cdot\vec{j}_{\text{ext}} = 0. \tag{11}$$

In this local form, we can formulate the following problem: assuming the external sources ρ_{ext} and j_{ext} to be known functions of space and time, determine the fields $\vec{E}$ and $\vec{B}$. As it stands, such problem is not properly formulated as there is no link between the sources and the fields. Therefore, one must supply an additional set of *constitutive relations* which generically can be written as

$$\vec{D} = \vec{D}(\vec{E},\vec{B}) \quad \text{and} \quad \vec{H} = \vec{H}(\vec{E},\vec{B}). \tag{12}$$

For simplicity, we will only consider the case where the above dependence is restricted to single fields

$$\vec{D} = \vec{D}(\vec{E}) \quad \text{and} \quad \vec{H} = \vec{H}(\vec{B}). \tag{13}$$

A priori, the functions (13) are unknown. However, we can assume that they are smooth and make a formal series expansion

$$\vec{D}(\vec{E}) = \vec{D}_0 + \left.\frac{\partial \vec{D}}{\partial \vec{E}}\right|_{\vec{E}=0} \vec{E} + \cdots \quad \text{and} \quad \vec{H}(\vec{B}) = \vec{H}_0 + \left.\frac{\partial \vec{H}}{\partial \vec{B}}\right|_{\vec{B}=0} \vec{B} + \cdots. \tag{14}$$

Truncating the series at linear order, the most general linear spacetime relations are written as

$$\vec{D}(t,x) = \int_{-\infty}^{\infty}\int_{\Omega} \bar{\varepsilon}(t,t';x,x')\vec{E}(t',x')\mathrm{d}x'\mathrm{d}t' \quad \text{and} \tag{15}$$

$$\vec{H}(t,x) = \int_{-\infty}^{\infty}\int_{\Omega} \bar{\mu}^{-1}(t,t';x,x')\vec{B}(t',x')\mathrm{d}x'\mathrm{d}t'. \tag{16}$$

Here $\bar{\varepsilon}$ and $\bar{\mu}$ are second rank, symmetric tensor densities characterising the response of a medium to the externally applied fields $\vec{E}$ and $\vec{B}$. In this form, we can interpret the fields $\vec{D}$ and $\vec{H}$ as averaged quantities over the medium. Thus, we can refer to $\vec{E}$ and $\vec{B}$ as the primary fields whilst $\vec{D}$ and $\vec{H}$ as the secondary fields.

Equations (15) and (16) allow us to define a local, homogeneous and isotropic media as those whose responses are given by

$$\vec{D}(t,x) = \int_{-\infty}^{\infty}\int_{\Omega} \varepsilon\, \delta(t-t';x-x')\vec{E}(t',x')\mathrm{d}t'\mathrm{d}x' = \varepsilon\vec{E}(t,x) \quad \text{and} \tag{17}$$

$$\vec{H}(t,x) = \int_{-\infty}^{\infty}\int_{\Omega} \mu^{-1}\, \delta(t-t';x-x')\vec{B}(t',x')\mathrm{d}t'\mathrm{d}x' = \frac{1}{\mu}\vec{B}(t,x), \tag{18}$$

where now ε and μ have become mere scalars. These are the simplest and most used forms for the electromagnetic constitutive relations. Note that, as they stand, they are very restrictive, we have made explicit each assumption leading to them.

Finally, equipped with the constitutive relations (17) and (18) let us tackle the problem stated earlier. Our aim is to obtain the fields $\vec{E}$ and $\vec{B}$ in terms of the external sources ρ_{ext} and $\vec{j}_{\text{ext}}$. Let us begin by noting that the six components of the fields $\vec{E}$ and $\vec{B}$ can be obtained in terms of four functions, the three components of the magnetic vector potential A and the scalar electric potential ϕ. Thus, from equations (7) and (8) it follows that

$$\vec{B} = \nabla \times \vec{A} \quad \text{and} \quad \vec{E} = -\nabla\phi - \frac{\partial}{\partial t}\vec{A}. \tag{19}$$

Using the constitutive relations (17) and (18) and substituting (19) into (9) and (10) we obtain a set of four coupled, linear, inhomogenous partial differential equations for the components of $\vec{A}$ and ϕ, namely

$$\nabla \times \nabla \times \vec{A} + \mu\varepsilon\frac{\partial^2}{\partial t^2}\vec{A} + \nabla\left[\mu\varepsilon\frac{\partial}{\partial t}\phi\right] = \mu\vec{j}_{\text{ext}} \quad \text{and} \tag{20}$$

$$\nabla\cdot\nabla\phi + \frac{\partial}{\partial t}\left[\nabla\cdot\vec{A}\right] = -\frac{\rho_{\text{ext}}}{\varepsilon}. \tag{21}$$

Space	Notation	Dimension	Coordinate basis
0-forms	$\Omega^0_{\mathcal{M}}$	1	$\{1\}$
1-forms	$\Omega^1_{\mathcal{M}}$	4	$\{\mathrm{d}x, \mathrm{d}y, \mathrm{d}z, \mathrm{d}t\}$
2-forms	$\Omega^2_{\mathcal{M}}$	6	$\{\mathrm{d}x \wedge \mathrm{d}y, \mathrm{d}y \wedge \mathrm{d}z, \mathrm{d}z \wedge \mathrm{d}x, \mathrm{d}x \wedge \mathrm{d}t, \mathrm{d}y \wedge \mathrm{d}t, \mathrm{d}z, \wedge \mathrm{d}t\}$
3-forms	$\Omega^3_{\mathcal{M}}$	4	$\{\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z, \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}t, \mathrm{d}x \wedge \mathrm{d}t \wedge \mathrm{d}z, \mathrm{d}t \wedge \mathrm{d}y \wedge \mathrm{d}z\}$
4-forms	$\Omega^4_{\mathcal{M}}$	1	$\{\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \wedge \mathrm{d}t\}$

Tabla 1: Spaces of p-forms over a four dimensional space $\mathcal{M}$

We can decouple this system by noting an extra freedom implicit in the definition of the potentials A and ϕ – equation (19) – that is, defining a new set of potentials

$$\vec{A}' = \vec{A} + \nabla\psi \quad \text{and} \quad \phi' = \phi - \frac{\partial}{\partial t}\psi, \tag{22}$$

where ψ is a differentiable scalar function, the same fields $\vec{E}$ and $\vec{B}$ are obtained. Therefore, substituting (51) into (20) and (21) and using the gauge condition

$$\nabla^2\psi - \mu\varepsilon\frac{\partial^2}{\partial t^2}\psi = 0, \tag{23}$$

one obtains the decoupled system of hyperbolic equations

$$\nabla^2\vec{A} - \mu\varepsilon\frac{\partial^2}{\partial t^2}\vec{A} = -\mu\vec{j}_{\text{ext}} \quad \text{and} \tag{24}$$

$$\nabla^2\phi - \mu\varepsilon\frac{\partial^2}{\partial t^2}\phi = -\frac{\rho_{\text{ext}}}{\varepsilon}, \tag{25}$$

which – together with a set of suitable boundary and initial conditions – allow us to uniquely determine the fields $\vec{E}$ and $\vec{B}$ in terms of the sources ρ_{ext} and $\vec{j}_{\text{ext}}$.

II | A GEOMETRIC FORMULATION OF ELECTROMAGNETISM

Electromagnetism can be formulated in modern geometric terms through the calculus of differential forms. There are numerous introductory texts to such geometric formalism (see (1; 2; 3; 4)). We urge the reader to explore text by Baldomir and Hammond (5) and the one by Gross and Kotiuga (6) for a thorough presentation of electromagnetism in such terms. Here, we shall re-write all of the content in the previous section in the differential form language so that the unfamiliar reader obtains a short dictionary for translating electromagnetism to geometry.

Roughly, in an n-dimensional space $\mathcal{M}$, a differential p-form ($0 \leq p \leq n$) is the argument of an integral over a p-dimensional domain. In this sense a 1-form is the argument of a line integral, a 2-form is the argument of a surface integral, and so forth. By definition a 0-form will simply be a scalar function. Algebraically, differential forms are totally antisymmetric multilinear maps acting on vector fields defined over the space $\mathcal{M}$. For each p they form a vector space denoted by $\Omega^p_{\mathcal{M}}$. In the present work, we will assume that the space $\mathcal{M}$ is four dimensional, accounting for three spatial and one time directions. Thus, over such a four dimensional space, we can accommodate these vector spaces as described in table 1.

In this sense, equations (1) and (2) are written as

$$\int_\gamma \vec{X}\cdot \mathrm{d}\ell = \int_\gamma (X_x \mathrm{d}x + X_y \mathrm{d}y + X_z \mathrm{d}z) = \int_\gamma \alpha \quad \text{and} \tag{26}$$

$$\int_\Sigma \vec{X}\cdot \mathrm{d}s = \int_\Sigma (X_z \mathrm{d}x\wedge \mathrm{d}y + X_y \mathrm{d}z\wedge \mathrm{d}x + X_x \mathrm{d}y\wedge \mathrm{d}z) = \int_\Sigma \beta, \tag{27}$$

with

$$\alpha \equiv X_x \mathrm{d}x + X_y \mathrm{d}y + X_z \mathrm{d}z \in \Omega^1_{\mathcal{M}} \tag{28}$$

and

$$\begin{aligned}\beta &\equiv X_z \mathrm{d}x\wedge \mathrm{d}y + X_y \mathrm{d}z\wedge \mathrm{d}x + X_x \mathrm{d}y\wedge \mathrm{d}z \\ &= \beta_{xy}\mathrm{d}x\wedge \mathrm{d}y + \beta_{zx}\mathrm{d}z\wedge \mathrm{d}x + \beta_{yz}\mathrm{d}y\wedge \mathrm{d}z \\ &= \sum_{i\neq j} \beta_{ij}\, \mathrm{d}x^i\wedge \mathrm{d}x^j \in \Omega^2_{\mathcal{M}},\end{aligned} \tag{29}$$

where

$$\beta_{ij} = \vec{X}\cdot \hat{e}_{(k)}. \tag{30}$$

The global expressions for Maxwell's equations can be summarised as

$$\oint_{\partial\Sigma^{(3)}} F = 0 \quad \text{and} \tag{31}$$

$$\oint_{\partial\Sigma^{(3)}} G = \int_{\Sigma^{(3)}} j, \tag{32}$$

where $\Sigma^{(3)} \subset \mathcal{M}$ is a three dimensional domain with boundary $\partial\Sigma^{(3)}$; and the fields $F, G \in \Omega^2_{\mathcal{M}}$ and the source $j \in \Omega^3_{\mathcal{M}}$ are defined as

$$F = \sum_{i=1}^{3} E_i\, \mathrm{d}x^i\wedge \mathrm{d}t + \sum_{i\neq j} B_{ij}\, \mathrm{d}x^i\wedge \mathrm{d}x^j, \tag{33}$$

$$G = \sum_{i=1}^{3} H_i\, \mathrm{d}x^i\wedge \mathrm{d}t + \sum_{i\neq j} D_{ij}\, \mathrm{d}x^i\wedge \mathrm{d}x^j \quad \text{and} \tag{34}$$

$$j = \rho_{\text{ext}}\mathrm{d}x\wedge \mathrm{d}y\wedge \mathrm{d}z + \sum_{i\neq j} j_{ij}\mathrm{d}x^i\wedge \mathrm{d}x^j\wedge \mathrm{d}t \tag{35}$$

where, as in (30), the components of the fluxes are given by

$$B_{ij} = \vec{B}\cdot \hat{e}_{(k)} \quad D_{ij} = \vec{D}\cdot \hat{e}_{(k)} \quad \text{and} \quad j_{ij} = \vec{j}_{\text{ext}}\cdot \hat{e}_{(k)}. \tag{36}$$

Again, the passing to from the global to the local form is a direct application of Stokes' Theorem,

$$\oint_{\partial\Sigma^{(3)}} F = \int_{\Sigma^{(3)}} \mathrm{d}F, \tag{37}$$

and equations (31) and (32) become

$$\mathrm{d}F = 0 \quad \text{and} \quad \mathrm{d}G = j. \tag{38}$$

These are the geometric expressions for Maxwell's equations. From the algebraic definition of the *exterior derivative* operator, one can easily obtain the partial differential equations for each of the fields

$\vec{E}$, $\vec{B}$, $\vec{D}$ and $\vec{H}$, equations (7) - (10). Moreover, noting that the exterior derivative is a *nilpotent* operator, i.e. $\mathrm{d}^2 = \mathrm{d} \circ \mathrm{d} = 0$, charge conservation [c.f. equation (11)] is simply a geometric identity, namely

$$\mathrm{d}j = \mathrm{d}\,[\mathrm{d}G] = \mathrm{d}^2 G = 0. \tag{39}$$

As before, we would like to obtain the field F in terms of the sources j. However, again, the sources are related to G but not to F. In order to correctly pose the problem, an additional link between G and F must be provided. Similarly, we will restrict our analysis to the most general linear expression of G in terms of F, that is, G is a convolution of F

$$G(p) = \int_\Omega \chi(p, p') F(p'), \tag{40}$$

where $\chi : T\binom{0}{2}\mathcal{M} \longrightarrow T\binom{0}{2}\mathcal{M}$ is a linear map called the *constitutive tensor* defining the response of the medium where F propagates.
In components, the constitutive tensor can be written as

$$\chi_{ab}^{\;\;cd} = \varepsilon_{abef}\kappa^{efcd}, \tag{41}$$

where ε_{abef} is the totally anti-symmetric Levi-Civita symbol and the indices of κ^{efcd} share some of the symmetries of a curvature tensor (7; 8). This is suggestive of a metric theory and, indeed, the simplest local constitutive relation [cf. equations (17) and (18)] is written as

$$G = \tilde{\kappa} \star_g F, \tag{42}$$

where $\star_g : \Omega^p_{\mathcal{M}} \longrightarrow \Omega^{n-p}_{\mathcal{M}}$ denotes the *Hodge* dual operator associated with a metric tensor defined over $\mathcal{M}$ [c.f. the pairing of dimensions in Table 1] and $\tilde{\kappa} = \sqrt{\varepsilon/\mu}$. Notice that, up to this point, it was not necessary to assume that the manifold $\mathcal{M}$ was equipped with a metric tensor g. The similarity of the constitutive tensor with the curvature of a Riemannian manifold served as a motivation to *introduce* such structure. Thus, the *metricity* assumption is made in addition to Maxwell's equations (38) and it plays a central role in the study of the symmetries of electromagnetic theory. Furthermore, the hyperbolic nature of the system of differential equations (24) and (25), stemming from the sign arrangement in (7) - (10), imposes an indefinite *Lorentzian* signature on g.
Continuing with our translation into geometric language, the homogeneous Maxwell equation, $\mathrm{d}F = 0$, implies the *local* existence of a class of potential 1-forms A such that

$$F = \mathrm{d}A = \mathrm{d}\,[A + \mathrm{d}\psi] \quad \text{with} \quad \psi : \mathcal{M} \longrightarrow \mathbb{R}. \tag{43}$$

Shortly we will deal with the scalar function (0-form) ψ. For the moment, it is easy to verify that

$$A = \phi\ \mathrm{d}t + \sum_{i=1}^{3} A_i\ \mathrm{d}x^i \in \Omega^1_{\mathcal{M}}, \tag{44}$$

yields the correct components [c.f. equation (19)] of the field form (33). Thus, using the constitutive relation (42) and the definition F in terms of the potential 1-form A, equation (43), and substituting into the inhomogeneous Maxwell's equations it follows that

$$\mathrm{d}G = \mathrm{d}\,[\tilde{\kappa} \star_g F] = \tilde{\kappa}\ \mathrm{d}\,[\star_g \mathrm{d}A] = j \in \Omega^3_{\mathcal{M}}, \tag{45}$$

or, equivalently, its dual equation

$$\tilde{\kappa}\ \star_g \mathrm{d} \star_g \mathrm{d}A = \star_g j \in \Omega^1_{\mathcal{M}}. \tag{46}$$

In a four dimensional (pseudo)Riemannian manifold (M,g), the co-differential, $\delta : \Omega^p_{\mathcal{M}} \longrightarrow \Omega^{p-1}_{\mathcal{M}}$, and the Laplace-deRham operator, $\Delta : \Omega^p_{\mathcal{M}} \longrightarrow \Omega^p_{\mathcal{M}}$ are defined as

$$\delta \equiv \star_g \mathrm{d} \star_g \quad \text{and} \quad \Delta \equiv \mathrm{d}\delta + \delta\mathrm{d}. \tag{47}$$

Thus, we can rewrite equation (46) as

$$\tilde{\kappa}\delta\mathrm{d}A = \tilde{\kappa}\,(\Delta A - \mathrm{d}\delta A) = \star_g j. \tag{48}$$

Notice that this equation does not depend on the representing member of the class of 1-form potentials, namely

$$\Delta A' - \mathrm{d}\delta A' = \Delta A - \mathrm{d}\delta A \quad \text{with} \quad A' = A + \mathrm{d}\psi. \tag{49}$$

As noted before, the 3-form j is a conserved current, that is, for any closed 3-hypersurface enclosing a volume Ω we have

$$\tilde{\kappa}\oint_{\partial\Omega} \star_g (\Delta A - \mathrm{d}\delta A) = \oint_{\partial\Omega} j = \int_{\Omega} \mathrm{d}j = 0. \tag{50}$$

In the case where the potential 1-form in the class is chosen so that the function ψ satsifies the condition

$$\Delta\psi = 0, \tag{51}$$

it follows that $\delta A = 0$ and $\tilde{\kappa}\star_g \Delta A$ coincides exactly with the conserved current. In such case, Maxwell's equations are simply written as

$$\tilde{\kappa}\star_g \Delta A = j. \tag{52}$$

Equation (51) is the geometric expression for the gauge condition (23) and the system (52) is equivalent to the pair of wave equations (24) and (25). Here, the Lorentzian signature of the metric tensor g guarantees the hyperbolicity of the Laplace-deRham operator, as opposed to its elliptic nature in the Riemannian case.

III | A SUMMARY OF THE VARIATIONAL BI-COMPLEX

In this section we take the next leap of abstraction in the formulation of Maxwell's electromagnetism. In section I we stated the empirical relations linking sources and fields and, through the aid of a constitutive relation and using Stokes' theorem, we obtained the local differential equations that one solves to obtain the fields in terms of the sources. When the equations of motion are empirically postulated it is useful to know if there is a variational formulation for them, that is, to know if the equations of motion are the conditions for stationary configurations of an action functional. This is the inverse problem of the calculus of variations. In the past decades, this problem has been formulated in the elegant framework of *jet-bundles* and the *variational bi-complex* (see (9)). The general idea is to use algebraic techniques for inverting well defined maps, rather than the standard variations, to obtain – if it exists – a Lagrangian density for a given set of equations of motion. In the present manuscript, we consider the standard problem to illustrate how the construction works. This section is based largely on the notes by Ian Anderson (9) and on the text by Peter Olver (10).
In the following, we shall construct various complexes. The vector calculus electromagnetic complex is

$$0 \longrightarrow C_{\mathbb{R}^3} \xrightarrow{\nabla} \mathrm{Vect}_{\mathbb{R}^3} \xrightarrow{\nabla\times} \mathrm{Vect}_{\mathbb{R}^3} \xrightarrow{\nabla\cdot} C_{\mathbb{R}^3} \longrightarrow 0, \tag{53}$$

where for each map

$$\mathrm{im}(\nabla) = \ker(\nabla\times) \quad \text{and} \quad \mathrm{im}(\nabla\times) = \ker(\nabla\cdot). \tag{54}$$

These maps are globally defined over $\mathbb{R}^3$ and (54) is the fact that we used to reformulate Maxwell's equations in terms of the vector and scalar potentials $\vec{A}$ and ϕ.
In section II we relaxed the topological structure of the domain were the fields are defined and exhibit the topological nature of Maxwell's electromagnetic theory. This time, we worked on a four dimensional differentiable manifold $\mathcal{M}$ with no other *a priori* structure defined on it. In this case, we can also observe the complex

$$0 \longrightarrow \Omega^0_{\mathcal{M}} \xrightarrow{\mathrm{d}_1} \Omega^1_{\mathcal{M}} \xrightarrow{\mathrm{d}_2} \Omega^2_{\mathcal{M}} \xrightarrow{\mathrm{d}_3} \Omega^3_{\mathcal{M}} \xrightarrow{\mathrm{d}_4} \Omega^4_{\mathcal{M}} \longrightarrow 0. \tag{55}$$

However, in this case

$$\mathrm{im}(\mathrm{d}_p) \subseteq \ker(\mathrm{d}_{p+1}) \tag{56}$$

Indicating us that the potential formulation is only locally valid.
Adding the constitutive relations, the electromagnetic sequence is

$$\begin{tikzcd}
\Omega^0_{\mathcal{M}} \arrow[r, "\mathrm{d}_1", "A' = A + \mathrm{d}\psi"'] & \Omega^1_{\mathcal{M}} \arrow[r, "\mathrm{d}_2", "F = \mathrm{d}A'"'] & \Omega^2_{\mathcal{M}} \arrow[d, "\star_g", "G = \tilde{\kappa}\star_g F"'] & & \\
& & \Omega^2_{\mathcal{M}} \arrow[r, "\mathrm{d}_3", "\mathrm{d}G = j"'] & \Omega^3_{\mathcal{M}} \arrow[r, "\mathrm{d}_4", "\mathrm{d}j = 0"'] & \Omega^4_{\mathcal{M}},
\end{tikzcd} \tag{57}$$

where A and A' are defined locally, i.e. in an open neighborhood $\mathcal{U}$ around each point $p \in \mathcal{M}$.
Note that, in our case, the role of the metric is crucial, allowing us to complete the sequence by mapping 2-forms.
We can build a top form by combining the elements belonging to the different spaces in the complex

$$\lambda = \frac{1}{2} F \wedge G + j \wedge A = \frac{1}{2}\tilde{\kappa}(\mathrm{d}A \wedge \star_g \mathrm{d}A) + j \wedge A. \tag{58}$$

We observe that this top-form is a functional of the unknown components of the potential 1-form A and the given sources, i.e. $\lambda = \lambda(A, \mathrm{d}A; j)$. In this exericise we have come to this conclusion. The variational formulation of Maxwell's equations consists in turning upside down our previous reasoining. That is, assuming that we have a four dimensional differentiable manifold equipped with a pseudo-Riemannian metric, i.e. considering the pair $(\mathcal{M}, g)$; and a *given* closed 3-form j, find the 1-form A such that the functional

$$S[A, \Omega] = \int_\Omega \lambda(A, \mathrm{d}A; j) = \int_\Omega \left[\frac{1}{2}\tilde{\kappa}(\mathrm{d}A \wedge \star_g \mathrm{d}A) + j \wedge A\right], \tag{59}$$

takes an extreme value. In this variational field theory problem, we are assuming from the beginning charge conservation by requiring that j be closed and, implicitly, the constitutive relation has to be metric.
The variational bi-complex is an extension of the complex (55) which allows us to turn the variational

problem stated above into another sequence. To this end, let us construct a fibre bundle[1] $\pi : E \longrightarrow \mathcal{M}$ with base manifold $(\mathcal{M}, g)$ and fibre $\mathcal{F}$ representing every possible value of the unknown functions (the components of the 1-form A) may have. Thus, we see that

$$\dim(E) = \dim(\mathcal{M}) + \dim(\mathcal{F}). \tag{60}$$

Thus, $\dim(E) = 4+4 = 8$ since there are four unknowns we are looking for. Let us consider a local section $s : \mathcal{M} \longrightarrow E$, whose local coordinates are given by

$$s(x) : x^a \longrightarrow [x^a, A^\alpha(x^a)] \quad \text{with} \quad \alpha = 1 \ldots 4, \tag{61}$$

the $k-$th order jet space of E is a fibration

$$\pi^k : J^k(E) \longrightarrow \mathcal{M} \tag{62}$$

whose local coordinates are naturally lifted as

$$\iota^k(s)(x) = \left[x^a, A^\alpha(x^a), A^\alpha_{i_1}, A^\alpha_{i_1 i_2} \ldots A^\alpha_{i_1,\ldots i_k}\right], \tag{63}$$

where $\iota^k(s) : \mathcal{M} \longrightarrow J^k(E)$ is a section of $J^k(E)$ and

$$A^\alpha_{i_1\ldots i_l} \equiv \frac{\partial^l}{\partial x^1 \partial x^2 \ldots \partial x^l} A^\alpha(x^a) \quad \text{for} \quad l = 0 \ldots k. \tag{64}$$

In the theory of differential equations, one usually considers the *infinite* jet bundle $J^\infty(E)$ consisting of all the jets of every order. Such space has the structure of a differentiable manifold and, therefore, we can consider the spaces of p-forms $\Omega^p\left[J^\infty(E)\right]$ defined on it.
In this setting, *contact forms* are defined as those whose action on horizontal vector fields of $J^\infty(E)$ vanish, that is, a differential form ω is a contact form if

$$[\iota^\infty(s)]^* \, \omega = 0, \tag{65}$$

where $[\iota^\infty(s)]^* : \Omega\left[J^\infty(E)\right] \longrightarrow \Omega(\mathcal{M})$ denotes the induced *pull-back* map.
Albeit abstract, contact forms are locally generated by

$$\theta^\alpha_{i_1\ldots i_k} = \mathrm{d}A^\alpha_{i_1\ldots i_k} - A^\alpha_{i_1\ldots i_k a}\mathrm{d}x^a. \tag{66}$$

That is,

$$\theta^\alpha = \mathrm{d}A^\alpha - A^\alpha_i \mathrm{d}x^i, \quad \theta^\alpha_i = \mathrm{d}A^\alpha_i - A^\alpha_{ij}\mathrm{d}x^j, \quad \theta^\alpha_{ij} = \mathrm{d}A^\alpha_{ij} - A_{ijk}\mathrm{d}x^k, \ldots \tag{67}$$

To see this, let us evaluate condition (65) for the fourth component of the potential 1-form A, that is

$$[\iota^\infty(s)]^* \, \theta^4 = [\iota^\infty(s)]^* \left(\mathrm{d}A^4 - A^4_i \mathrm{d}x^i\right) = \mathrm{d}\phi(x^a) - \frac{\partial}{\partial x^i}\phi(x^a)\, \mathrm{d}x^i = 0, \tag{68}$$

which is simply the definition of the exterior derivative of the scalar potential $\phi : \mathcal{M} \longrightarrow \mathbb{R}$. However, note that the vanishing of the left hand side of equation (68) only occurs when restricted to $\mathcal{M}$. In general, A^α and A^α_i denote independent coordinates in the jet space $J^\infty(E)$ which can vary freely. Understanding and appreciating this fact is the essence of William Burke's dedication of his book (2).
Following Anderson (9), let us introduce a new type of forms on the jet space. We will say a form $\omega \in \Omega^p\left[J^\infty(E)\right]$ is of type (r, s) if its action on p vector fields

$$\omega(X_{(1)}, \ldots, X_{(p)}) = 0 \tag{69}$$

[1] See the text by Nash and Zen for an introduction to the theory of fibre bundles (11) or the foundational book by Kobayashi and Nomizu (12) for a formal and complete approach

whenever more than s of the vectors are vertical, or more than r of the vectors annihilate all the contact 1-forms. Thus, we can decompose $\Omega^p\left[J^\infty(E)\right]$, the space of p-forms over the infinite jet, into two independent classes, namely, horizontal and vertical forms. Horizontal forms correspond to the usual generators of the cotangent bundle of $\mathcal{M}$, while vertical ones are all the contact forms such that

$$\Omega^p\left[J^\infty(E)\right] = \bigoplus_{r+s=p} \Omega^{r,s}\left[J^\infty(E)\right], \tag{70}$$

where $\Omega^{r,s}\left[J^\infty(E)\right]$ is denotes the space of forms of type (r,s) on the jet.
Now, the exterior derivative operator defined over the jet space splits into two parts, $\mathrm{d} = \mathrm{d}_H + \mathrm{d}_V$, where

$$\mathrm{d}_H : \Omega^{r,s}\left[J^\infty(E)\right] \longrightarrow \Omega^{r+1,s}\left[J^\infty(E)\right] \quad \text{and} \tag{71}$$

$$\mathrm{d}_V : \Omega^{r,s}\left[J^\infty(E)\right] \longrightarrow \Omega^{r,s+1}\left[J^\infty(E)\right]. \tag{72}$$

To see the coordinate expression of the operators (71) and (72), let us consider a function $f : J^\infty(E) \longrightarrow \mathbb{R}$ written as $f = f(x^a, A^\alpha, A^\alpha_i, A^\alpha_{ij}, \ldots)$, the horizontal derivative of f is expressed as

$$\mathrm{d}_H f = \left[\frac{\partial}{\partial x^i} + A^\alpha_i \frac{\partial}{\partial A^\alpha} + A^\alpha_{ij}\frac{\partial}{\partial A^\alpha_j} + \ldots\right](f)\mathrm{d}x^i = D_i(f)\mathrm{d}x^i, \tag{73}$$

while the vertical is

$$\mathrm{d}_V f = \frac{\partial f}{\partial A^\alpha}\theta^\alpha + \frac{\partial f}{\partial A^\alpha_i}\theta^\alpha_i + \ldots \tag{74}$$

In this construction, we can see that motions in the vertical directions can be interpreted as the usual variations of the potentials in the calculus of variations. Here, those are simply vertical exterior derivatives. Thus, in a four dimensional manifold $\mathcal{M}$, we can depict the variational bi-complex as a vertical extension of the de Rham complex (55) of the form

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \vdots & \vdots & \\
0 \arrow[r] & \Omega^{0,2} \arrow[r, "\mathrm{d}_H"] \arrow[u, "\mathrm{d}_V"] & \Omega^{1,2} \arrow[r, "\mathrm{d}_H"] \arrow[u, "\mathrm{d}_V"] & \Omega^{2,2} \arrow[r, "\mathrm{d}_H"] \arrow[u, "\mathrm{d}_V"] & \Omega^{3,2} \arrow[r, "\mathrm{d}_H"] \arrow[u, "\mathrm{d}_V"] & \Omega^{4,2} \arrow[r] \arrow[u, "\mathrm{d}_V"] & 0 \\
0 \arrow[r] & \Omega^{0,1} \arrow[r, "\mathrm{d}_H"] \arrow[u, "\mathrm{d}_V"] & \Omega^{1,1} \arrow[r, "\mathrm{d}_H"] \arrow[u, "\mathrm{d}_V"] & \Omega^{2,1} \arrow[r, "\mathrm{d}_H"] \arrow[u, "\mathrm{d}_V"] & \Omega^{3,1} \arrow[r, "\mathrm{d}_H"] \arrow[u, "\mathrm{d}_V"] & \Omega^{4,1} \arrow[r] \arrow[u, "\mathrm{d}_V"] & 0 \\
0 \arrow[r] & \Omega^{0,0} \arrow[r, "\mathrm{d}_H"] \arrow[u, "\mathrm{d}_V"] & \Omega^{1,0} \arrow[r, "\mathrm{d}_H"] \arrow[u, "\mathrm{d}_V"] & \Omega^{2,0} \arrow[r, "\mathrm{d}_H"] \arrow[u, "\mathrm{d}_V"] & \Omega^{3,0} \arrow[r, "\mathrm{d}_H"] \arrow[u, "\mathrm{d}_V"] & \Omega^{4,0} \arrow[r] \arrow[u, "\mathrm{d}_V"] & 0.
\end{tikzcd} \tag{75}$$

Note that the decomposition (70) of $\Omega^p\left[J^\infty(E)\right]$ corresponds to the various diagonals of the variational bi-complex (**??**).
The electromagnetic top form (58) can be promoted to an element of $\tilde{\lambda} \in \Omega^{4,0}\left[J^\infty(E)\right]$ and we can write

the action functional (59) as

$$S[\lambda,\Omega] = \int_\Omega [\iota^\infty(s)]^*(\tilde{\lambda}) = \int_\Omega \lambda(A,\mathrm{d}A;j)$$
$$= \int_\Omega L\left[A^\alpha(x^a), \frac{\partial}{\partial x^b}A^\alpha(x^a);\ j(x^a)\right]\mathrm{d}x^1\wedge\mathrm{d}x^2\wedge\mathrm{d}x^3\wedge\mathrm{d}x^4. \tag{76}$$

We will refer to the form $\tilde{\lambda}$ as the electromagnetic *Lagrangian*, where the components and derivatives of the potential 1-form A are free variables.
Thus far this construction has been very abstract, demanding a huge amount of differential equations theory to tackle a seemingly standard problem. Here comes the conceptual advantage of the construction of the variational bi-complex. Taking exterior derivatives is a simple algebraic exercise and, for any Lagrangian $\Omega^{n,0}[J^\infty(E)]$, its vertical exterior derivative is

$$\mathrm{d}_V\tilde{\lambda} = E(\tilde{\lambda}) + \mathrm{d}_H\eta \quad \text{for some} \quad \eta\in\Omega^{n-1,1}[J^\infty(E)], \tag{77}$$

where E denotes the *Euler-Lagrange* operator defined as

$$E(\tilde{\lambda}) = \sum_{\alpha=1}^{\dim(\mathcal{F})} E_\alpha(L)\theta^\alpha\wedge\mathrm{d}x^1\wedge\cdots\wedge\mathrm{d}x^n \in \Omega^{(n,1)}[J^\infty(E)]. \tag{78}$$

In the present case, the components of the Euler-Lagrange operator are simply

$$E_\alpha(L) = \left[\frac{\partial}{\partial A^\alpha} - D_i\frac{\partial}{\partial A^\alpha_i}\right]L \tag{79}$$

which are the usual field theory Euler-Lagrange equations.
Notice, however, that our geometric Lagrangian input is the form $\tilde{\lambda}$, and not directly the function L. In this sense, our equations of motion have to be obtained from (77), i.e.

$$E(\tilde{\lambda}) = \mathrm{d}_V\tilde{\lambda} - \mathrm{d}_H\eta. \tag{80}$$

We need to eliminate the *exact* horizontal form $\mathrm{d}_H\eta$. This is usually done through *integration by parts* and demanding that the filed variations vanish in the boundary. In the geometric setting this is achieved by introducing a *co-augmentation* map

$$I:\Omega^{n,s}[J^\infty(E)] \longrightarrow \Omega^{n,s}[J^\infty(E)], \tag{81}$$

such that

$$I(\mathrm{d}_H\eta) = 0 \quad \text{for any} \quad \eta\in\Omega^{n-1,1}[J^\infty(E)], \tag{82}$$

any horizontal top form $\omega\in\Omega^{n,s}[J^\infty(E)]$ can be written as

$$\omega = I(\omega) + \mathrm{d}_H\eta \tag{83}$$

and I can be seen as a projector, that is

$$I^2 = I. \tag{84}$$

Therefore, the Euler-Lagrange equations for a given Lagrangian form $\tilde{\lambda}$ are written as the vanishing of the Euler-Lagrange operator, i.e.

$$E(\tilde{\lambda}) = I(\mathrm{d}_V\tilde{\lambda} - \mathrm{d}_H\eta) = I(\mathrm{d}_V\tilde{\lambda}) = 0. \tag{85}$$

IV | DERIVING MAXWELL EQUATIONS THROUGH THE VARIATIONAL BI-COMPLEX

In this section we present the use of the variational bi-complex in deriving Maxwell equations using the computer algebra system Maple™ together with its built-in DifferentialGeometry, Tensor and JetCalculus libraries (see (13)). Some familiarity with the Maple™ language is assumed. Also, we encourage the reader to explore the very well written documentation of these libraries. We will break the script into a preamble, definition of geometric objects and the calculation using the variational bi-complex.

1 | Preamble

Firstly we restart the Maple™ kernel by

```
> restart:
```

We need to call the libraries we will use through our calculation, define the independent and dependent variables. Since we want to have certain freedom on the dimension of the base the base manifold $\mathcal{M}$ we will count the number of independent variables through the *nops* command. The dependent variables will be included into an array for ease of further manipulation from the very beginning. Note the use of sequences, concatenations and counters. This is a very convenient way to generate variables in a generic code

```
> with(DifferentialGeometry); with(Tensor); with(Tools); with(JetCalculus); with(PDEtools)

> Preferences("TensorDisplay", 1); Preferences("PrettyPrint", false); Preferences("JetNotation",
"JetNotation1")
> vars := [x, y, z, t]; n := nops(vars)
> A__def := [seq(cat(A__, i), i = 1 .. n-1), phi]
```

Now we have all the ingredients to define the jet space. To this end, we will construct $J^2(E)$ using the DGsetup calling of the DifferentialGeometry library. The first argument corresponds to the base manifold coordinates, the second to the fibre coordinates, the third is the name of the space, the fourth is the order of the jet space and, finally, verbose prints the protected variables for the rest of the script:

```
> DGsetup(vars, A__def, Maxwell, 2, verbose)
```

'The following coordinates have been protected:'

[x, y, z, t, A__1[], A__2[], A__3[], phi[], A__1[1], A__1[2], . . . , A__3[4, 4], phi[1, 1], phi[1, 2], phi[1, 3], phi[1, 4], phi[2, 2], phi[2, 3], phi[2, 4], phi[3, 3], phi[3, 4], phi[4, 4]]

'The following vector fields have been defined and protected:'

[D_x,D_y,D_z,D_t,D_A__1[],. . . ,D_ phi[3, 4]]

'The following differential 1-forms have been defined and protected:'

[dx, dy, dz, dt, dA__1[], dA__2[], . . . ,dphi[4, 4]]

'The following type [1,0] biforms have been defined and protected::'

[Dx, Dy, Dz, Dt]

'The following type [0,1] biforms (contact 1-forms) have been defined and protected::'

[CA__1[], CA__2[], ..., Cphi[4, 4]]

'frame name: Maxwell'

Maple™ denotes with a capital D the basis vectors of for the tangent bundle of the base manifold $\mathcal{M}$, e.g. $D\_x = \dfrac{\partial}{\partial x}$, but it uses a similar notation for $\Omega^{1,0}[J^\infty(E)]$, e.g. Dx. Contact forms are denoted with a C, e.g. $CA\_\_1[] = \mathrm{d}A^1 - A^1_i\,\mathrm{d}x^i$. Finally, notice that this simple lines has generated a basis for each of the spaces in the three bottom rows of the variational bi-complex (**??**).
The amount of locked variables is sufficiently large and their syntax quite elaborate. Thus, it will be convenient to 'bundle' all this information into various arrays. This will greatly simplify the rest of the script. To this end, we will use the *DGinfo* command, contained in the *Tools* library

```
> dQ := DGinfo("FrameBaseForms"): DQ := DGinfo("FrameBaseVectors"):
> dX := DGinfo(Maxwell, "FrameHorizontalBiforms"):
```

Since we are considering the base as the spacetime pseudo-Riemannian manifold $(\mathcal{M}, g)$ and we will use standard units, some care must be taken with speed of light c pre-factors for the time components of the metric. Thus, we redefine the 4th component using the *evalDG* command of the DifferentialGeometry library, telling Maple™ that we are defining a geometric quantity

```
> dQ[n] := evalDG(c*dQ[n]); DQ[n] := evalDG(DQ[n]/c); dX[n] := evalDG(c*dX[n])
```

Finally, we will generate an array containing all the base 2 and 3-forms

```
> TwoForm := GenerateForms(dQ, 2); ThreeForm := GenerateForms(dQ, 3):
```

2 | Geometric setting: metric, potential and current

Now we can start with the construction of Maxwell's electromagnetic theory. We start this exercise by assuming that the base manifold $\mathcal{M}$ is equipped with a pseudo-Riemannian metric. We are free to use any metric tensor for $\mathcal{M}$. For simplicity, we will assume a Minkowski spacetime

```
> g := evalDG(sum(`t`(dQ[i], dQ[i]), i = 1 .. n-1)-`t`(dQ[n], dQ[n]));
> g__inv := InverseMetric(g):
```

$$g := dx \otimes dx + dy \otimes dy + dz \otimes dz - c^2 dt \otimes dt$$

Here, 't' indicates tensor product.
The potential 1-form is built as an element of $\Omega^{1,0}[J^\infty(E)]$

```
> A__comp := [A__1[], A__2[], A__3[], phi[]]
> A := evalDG(sum(A__comp[i]*dX[i], i = 1 .. n-1)-A__comp[n]*dX[n]/c)
```

$$A := A_1[\,]Dx + A_2[\,]Dy + A_3[\,]Dz - \phi[\,]Dt$$

Here, the minus sign before the scalar potential is explicitly used so that the sign convention in the electric field expression in (19) is satisfied (c.f. the components of F, below).
The current 3-form is defined as

```
> j__comp := [j__1(vars), j__2(vars), j__3(vars), rho(vars)]
> j__s := evalDG(TwoForm[1]*j__comp[3]-TwoForm[2]*j__comp[2]+TwoForm[4]*j__comp[1])
> rho__v := evalDG(j__comp[n]*ThreeForm[1])
> j := convert(evalDG(-I*(rho__v-`w`(j__s/c, dQ[n]))), DGbiform)[n]
```

$$j := -\mathrm{I}\rho Dx \wedge Dy \wedge Dz + \mathrm{I} j_3 Dx \wedge Dy \wedge Dt - \mathrm{I} j_2 Dx \wedge Dz \wedge Dt + \mathrm{I} j_1 Dy \wedge Dz \wedge Dt$$

Note that we first build the 3-form (35), whose components are functions of the independent variables, and then we use the *convert* command with the argument *DGbiform* to turn it into an element of $\Omega^{3,0}[J^\infty(E)]$. This is all the input of the problem.

3 | Exterior derivatives and Hodge star: fields and constitutive relation

The field form F is simply the horizontal exterior derivative of the potential $(1,0)$-form A, defined above

```
> F := HorizontalExteriorDerivative(A)
```

$$F := -(A_{2_1} + A_{1_2})Dx \wedge Dy - (A_{3_1} + A_{1_3})Dx \wedge Dz - (-A_{3_2} + A_{2_3})Dy \wedge Dz - (\phi_1 + A_{1_4})Dx \wedge Dt - (\phi_2 + A_{2_4})Dy \wedge Dt - (\phi_3 + A_{3_4})Dz \wedge Dt$$

Compare with the definition of the 2-form F in terms of the components of the fields $\vec{E}$ and $\vec{B}$, equation (33), and their corresponding expressions in terms of the vector and scalar potentials, equation (19).
Here we will make the assumption of the metric constitutive relation (42). However, the field form defined above is a (2,0)-form. In order to perform the calculation we turn F into a 2-form over $(\mathcal{M}, g)$, apply the Hodge dual operator and turn the result back into a (2,0)-form

```
> G := convert(simplify(HodgeStar(g, convert(sqrt(epsilon/mu)*F, DGform)), symbolic), DGbiform)[3
```

These are all the geometric objects we need for our variational problem.

4 | Field Equations

To obtain the required field equations, we need to construct the (4,0)-form (58). Here, due to the signature of the metric, we use a purely imaginary Lagrangian form

```
> lambda := evalDG(I*((1/2)*`w`(F, G)+`w`(j, A)))
```

Since we already defined the field (2,0)-forms in terms of the potential, the output of this code line is the explicit form of the function L in (76).
Finally, the field equations are written simply as the vanishing of each component of the Euler-Lagrange operator applied to the (4,0)-form λ, above. Thus, using the expression for $E(\tilde{\lambda})$, equation (85), we have

```
> simplify(IntegrationByParts(VerticalExteriorDerivative(lambda)), size)
```

$$\frac{-1}{\sqrt{\mu}c}(-j_1 c\sqrt{\mu}+((A_{1_{2,2}}+A_{1_{3,3}}-A_{2_{1,2}}-A_{3_{1,3}})c^2-\phi_{1,4}-A_{1_{4,4}})\sqrt{\varepsilon})Dx\wedge Dy\wedge Dz\wedge Dt\wedge CA_1[\,]+$$
$$\frac{1}{\sqrt{\mu}c}(j_2 c\sqrt{\mu}+((A_{1_{1,2}}-A_{2_{1,1}}-A_{2_{3,3}}+A_{3_{2,3}})c^2+\phi_{2,4}+A_{2_{4,4}})\sqrt{\varepsilon})Dx\wedge Dy\wedge Dz\wedge Dt\wedge CA_2[\,]+$$
$$\frac{1}{\sqrt{\mu}c}(j_3 c\sqrt{\mu}+((A_{1_{1,3}}+A_{2_{2,3}}-A_{3_{1,1}}-A_{3_{2,2}})c^2+\phi_{3,4}+A_{3_{4,4}})\sqrt{\varepsilon})Dx\wedge Dy\wedge Dz\wedge Dt\wedge CA_3[\,]-$$
$$\frac{1}{\sqrt{\mu}c}(\rho c\sqrt{\mu}-(\phi_{1,1}+\phi_{2,2}+\phi_{3,3}+A_{1_{1,4}}+A_{2_{2,4}}+A_{3_{3,4}})\sqrt{\varepsilon})Dx\wedge Dy\wedge Dz\wedge Dt\wedge Cphi[\,]$$

which, using the fact that $c^2 = \varepsilon\mu$, we observe that we have recovered the coupled Maxwell equations for the vector and scalar potentials $\vec{A}$ and ϕ [c.f. equations (20) and (21)], respectively. This completes our task of obtaining Maxwell Field Equations from an action principle using the variational bi-complex. It is a programing exercise to implement the gauge condition to decouple the equations and recover (24) and (25), i.e. equation (52).

V | CLOSING REMARKS

In this manuscript we have gone from the empirically obtained Maxwell integral relations, to their local vector calculus formulation. We have recalled the origin of the constitutive relations and use the simplest class to close the field theory problem, namely, given the sources obtain the fields. Then, we note that the homogeneous vector equations are mere geometric identities which allows us to express the fields $\vec{E}$ and $\vec{B}$ in terms of potentials. Finally, we closed the first section with the standard wave equations for the potentials. Then, in Section II, we carried out the same analysis in the geometric setting of differentiable manifolds, differential forms and we have introduce the metric tensor as an ingredient to express the constitutive relation. We tried to follow the same reasoning so that the unfamiliar reader can use such section as a dictionary between the vector and the geometric formulation of Maxwell's theory. In Section III, motivated by the topological sequences of the nabla and exterior derivative operators, we sketched the construction of the variational bi-complex used to geometrise the standard calculus of variations. Finally, in spite of the abstract formulation of the Euler-Lagrange equations in terms of vertical exterior derivatives and an integration by parts operator, we presented a simple Maple™ script to directly obtain Maxwell's equations in terms of the components of the magnetic vector and electric scalar potentials. In writing the Maple™ script, we have highlighted explicitly the places where our assumptions about the construction of the theory were made. Thus, it is our hope that the reader can modify and play with those, wether increasing the number of dimensions, or changing the metric tensor, or the constitutive relations, or relaxing these assumptions altogether and explore how far can she go.

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